

# ON TITCHMARSH-WEYL FUNCTIONS OF FIRST-ORDER SYMMETRIC SYSTEMS WITH ARBITRARY DEFICIENCY INDICES

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**ABSTRACT.** We study general (not necessarily Hamiltonian) first-order symmetric systems  $Jy'(t) - B(t)y(t) = \Delta(t)f(t)$  on an interval  $[a, b)$  with the regular endpoint  $a$ . The deficiency indices  $n_{\pm}$  of the corresponding minimal relation  $T_{min}$  may be arbitrary (possibly unequal). Our approach is based on the concept of a decomposing boundary triplet, which enables one to parametrize various classes of extensions of  $T_{min}$  (self-adjoint,  $m$ -dissipative, etc.) in terms of boundary conditions imposed on regular and singular values of a function  $y \in \text{dom } \mathcal{T}_{max}$  at the endpoints  $a$  and  $b$  respectively. In particular, we describe self-adjoint and  $\lambda$ -depending Nevanlinna boundary conditions which are analogs of separated ones for Hamiltonian systems. With a boundary value problem involving such conditions we associate the  $m$ -function  $m(\cdot)$ , which is an analog of the Titchmarsh-Weyl coefficient for the Hamiltonian system. In the simplest case of minimal (unequal) deficiency indices  $n_{\pm}$  the  $m$ -function  $m(\cdot)$  coincides with the rectangular Titchmarsh-Weyl coefficient introduced by Hinton and Schneider. We parametrize all  $m$ -functions in terms of the Nevanlinna boundary parameter at the endpoint  $b$  by means of the formula similar to the known Krein formula for resolvents. Application of these results to differential operators of an odd order enables us to complete the results by Everitt and Krishna Kumar on the Titchmarsh-Weyl theory of such operators.

## 1. INTRODUCTION

Assume that  $H$  and  $\hat{H}$  are finite dimensional Hilbert spaces with  $\dim H = \nu_+$  and  $\dim \hat{H} = \hat{\nu}$  and let

$$(1.1) \quad H_0 = H \oplus \hat{H}, \quad \mathbb{H} = H_0 \oplus H = H \oplus \hat{H} \oplus H.$$

The main object of the paper is a first-order symmetric system of differential equations defined on an interval  $\mathcal{I} = [a, b)$ ,  $-\infty < a < b \leq \infty$ , with the regular endpoint  $a$  and singular, generally speaking, endpoint  $b$ . Such a system is of the form [1, 14]

$$(1.2) \quad Jy'(t) - B(t)y(t) = \Delta(t)f(t), \quad t \in \mathcal{I},$$

where  $B(t) = B^*(t)$  and  $\Delta(t) \geq 0$  are the  $[\mathbb{H}]$ -valued functions on  $\mathcal{I}$  and

$$(1.3) \quad J = \begin{pmatrix} 0 & 0 & -I_H \\ 0 & iI_{\hat{H}} & 0 \\ I_H & 0 & 0 \end{pmatrix} : H \oplus \hat{H} \oplus H \rightarrow H \oplus \hat{H} \oplus H.$$

We suppose that the system (1.2) is definite, that is for each  $\lambda \in \mathbb{C}$  the equalities

$$(1.4) \quad Jy'(t) - B(t)y(t) = \lambda\Delta(t)y(t)$$

and  $\Delta(t)y(t) = 0$  a.e. on  $\mathcal{I}$  yield  $y(t) = 0$ ,  $t \in \mathcal{I}$ .

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The system (1.2) is called Hamiltonian if  $\widehat{H} = \{0\}$ , in which case

$$(1.5) \quad J = \begin{pmatrix} 0 & -I_H \\ I_H & 0 \end{pmatrix} : H \oplus H \rightarrow H \oplus H.$$

Assume that  $\mathcal{L}_\Delta^2(\mathcal{I})$  is the semi-Hilbert space of  $\mathbb{H}$ -valued functions  $f(t)$  on  $\mathcal{I}$  with  $\|f\|_\Delta^2 := \int_{\mathcal{I}} (\Delta(t)f(t), f(t))_{\mathbb{H}} dt < \infty$ ,  $L_\Delta^2(\mathcal{I})$  is the corresponding Hilbert space of equivalence classes,  $\pi$  is a quotient map from  $\mathcal{L}_\Delta^2(\mathcal{I})$  onto  $L_\Delta^2(\mathcal{I})$  and  $\tilde{\pi} = \pi \oplus \pi$ . Denote also by  $\mathcal{L}_\Delta^2[\mathcal{K}, \mathbb{H}]$  the set of all operator functions  $Y(t) (\in [\mathcal{K}, \mathbb{H}])$  on  $\mathcal{I}$  such that  $Y(t)h \in \mathcal{L}_\Delta^2(\mathcal{I})$  for each  $h \in \mathcal{K}$  (here  $\mathcal{K}$  is a finite dimensional Hilbert space).

As is known the extension theory of symmetric linear relations is the natural approach to boundary value problems involving symmetric systems (see [39, 28, 7, 8, 17, 23, 2, 29] and references therein). According to [39] the system (1.2) generates linear relations  $\mathcal{T}_{min}$  and  $\mathcal{T}_{max}$  in  $\mathcal{L}_\Delta^2(\mathcal{I})$  and minimal and maximal relations  $T_{min} = \tilde{\pi}\mathcal{T}_{min}$  and  $T_{max} = \tilde{\pi}\mathcal{T}_{max}$  in  $L_\Delta^2(\mathcal{I})$ . It turns out that  $T_{min}$  is a closed symmetric relation with not necessarily equal deficiency indices  $n_\pm$  and  $T_{max} = T_{min}^*$ . Moreover, the equality

$$(1.6) \quad [y, z]_b = \lim_{t \uparrow b} (Jy(t), z(t)), \quad y, z \in \text{dom } \mathcal{T}_{max},$$

defines a skew-Hermitian bilinear form on the domain of  $\mathcal{T}_{max}$  with finite indices of inertia  $\nu_{b+}$  and  $\nu_{b-}$ .

A description of various classes of extensions of  $T_{min}$  (self-adjoint,  $m$ -dissipative, etc.) in terms of boundary conditions is an important problem in the spectral theory of symmetric systems. In particular, a boundary value problem for the system (1.2) with self-adjoint separated boundary conditions generates the Fourier transform with the spectral function of the minimal dimension. Assume that the system (1.2) is Hamiltonian,  $n_+ = n_- =: n$  and let  $y(t) = \{y_0(t), y_1(t)\} (\in H \oplus H)$  be the representation of a function  $y \in \text{dom } \mathcal{T}_{max}$ . Then according to [19] the general form of self-adjoint separated boundary conditions is

$$(1.7) \quad \cos B_1 y_0(a) + \sin B_1 y_1(a) = 0, \quad [y, \chi_j]_b = 0, \quad j = 1 \div \nu_b, \quad y \in \text{dom } \mathcal{T}_{max},$$

where  $B_1 = B_1^* \in [H]$ ,  $\nu_b = n - \dim H$  and  $\chi_1, \chi_2, \dots, \chi_{\nu_b}$  are linearly independent modulo  $\text{dom } \mathcal{T}_{min}$  functions from  $\text{dom } \mathcal{T}_{max}$  such that  $\chi_j(0) = 0$  and  $[\chi_j, \chi_k]_b = 0$ ,  $j \neq k$ . An element  $y_b := \{[y, \chi_j]_b\}_1^{\nu_b} \in \mathbb{C}^{\nu_b}$  is called a singular boundary value of a function  $y \in \text{dom } \mathcal{T}_{max}$ . Observe that for differential operators the notion of a singular boundary value as well as formula (1.7) go back to the paper by Calkin [3] (see also [9, Ch.13.2]).

Boundary conditions (1.7) generate a self-adjoint extension  $\tilde{A}$  of  $T_{min}$  given by  $\tilde{A} = \tilde{\pi}\{\{y, f\} \in \mathcal{T}_{max} : y \text{ satisfies (1.7)}\}$ . The resolvent of  $\tilde{A}$  is defined by  $(\tilde{A} - \lambda)^{-1}\tilde{f} = \pi y_f$ , where  $y_f$  is the  $\mathcal{L}_\Delta^2$ -solution of the boundary problem involving the system

$$(1.8) \quad Jy'(t) - B(t)y(t) = \lambda\Delta(t)y + \Delta(t)f(t), \quad f \in \tilde{f}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and the boundary conditions (1.7). Moreover, according to [19] the Titchmarsh - Weyl coefficient  $M_{TW}(\lambda) (\in [H])$  of the boundary problem (1.8), (1.7) is defined by the relations

$$(1.9) \quad v(t, \lambda) := \varphi(t, \lambda)M_{TW}(\lambda) + \psi(t, \lambda) \in \mathcal{L}_\Delta^2[H, \mathbb{H}] \quad \text{and} \quad [v(\cdot, \lambda)h, \chi_j]_b = 0, \quad h \in H.$$

for all  $j = 1 \div \nu_b$ . Here  $\varphi(\cdot, \lambda)$  and  $\psi(\cdot, \lambda)$  are the  $[H, \mathbb{H}]$ -valued operator solutions of Eq. (1.4) with the initial data  $\varphi(a, \lambda) = (\sin B_1 : -\cos B_1)^\top$  and  $\psi(a, \lambda) = (-\cos B_1 : \sin B_1)^\top$ . Note also the paper [26], in which the Titchmarsh - Weyl coefficient is defined by means of a limiting process from a compact interval  $[a, \beta] \subset \mathcal{I}$ . It turns out that  $M_{TW}(\cdot)$  is a Nevanlinna operator function, i.e.,  $M_{TW}(\cdot)$  is holomorphic on  $\mathbb{C} \setminus \mathbb{R}$  and  $\text{Im} \lambda \cdot \text{Im} M_{TW}(\lambda) \geq$

0,  $M_{TW}^*(\lambda) = M_{TW}(\bar{\lambda})$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Moreover, the spectral function of  $M_{TW}(\cdot)$  is a spectral function of the corresponding Fourier transform with the minimal dimension.

Another approach to description of boundary conditions is based on the concept of a decomposing boundary triplet for  $T_{max}$  (see [38] for symmetric systems and [35, 36, 37] for differential operators). To explain this concept note that there exist finite-dimensional Hilbert spaces  $\mathcal{H}_b$  and  $\widehat{\mathcal{H}}_b$  and a surjective linear map

$$(1.10) \quad \Gamma_b = (\Gamma_{0b} : \widehat{\Gamma}_b : \Gamma_{1b})^\top : \text{dom } \mathcal{T}_{max} \rightarrow \mathcal{H}_b \oplus \widehat{\mathcal{H}}_b \oplus \mathcal{H}_b$$

such that the bilinear form (1.6) admits the representation

$$(1.11) \quad [y, z]_b = i \cdot \text{sign}(\nu_{b+} - \nu_{b-})(\widehat{\Gamma}_b y, \widehat{\Gamma}_b z) - (\Gamma_{1b} y, \Gamma_{0b} z) + (\Gamma_{0b} y, \Gamma_{1b} z).$$

Moreover, let  $X_a \in [\mathbb{H}]$  be the operator such that  $X_a^* J X_a = J$ , and let

$$\Gamma_a = (\Gamma_{0a} : \widehat{\Gamma}_a : \Gamma_{1a})^\top : AC(\mathcal{I}; \mathbb{H}) \rightarrow H \oplus \widehat{H} \oplus H.$$

be the block representation of the linear map  $\Gamma_a y = X_a y(a)$ ,  $y \in AC(\mathcal{I}; \mathbb{H})$  (here  $AC(\mathcal{I}; \mathbb{H})$  is the set of all absolutely continuous  $\mathbb{H}$ -valued functions on  $\mathcal{I}$ ). By using  $\mathcal{H}_b$ ,  $\widehat{\mathcal{H}}_b$  and  $\Gamma_a$ ,  $\Gamma_b$  one constructs the Hilbert space  $\mathcal{H}_0$ , the subspace  $\mathcal{H}_1$  in  $\mathcal{H}_0$  and the linear maps  $\Gamma'_j : \text{dom } \mathcal{T}_{max} \rightarrow \mathcal{H}_j$ ,  $j \in \{0, 1\}$ , such that the classical Lagrange's identity takes the form

$$(1.12) \quad (f, z)_\Delta - (y, g)_\Delta = (\Gamma'_1 y, \Gamma'_0 z) - (\Gamma'_0 y, \Gamma'_1 z) + i \text{sign}(n_+ - n_-)(P_2 \Gamma'_0 y, P_2 \Gamma'_0 z)$$

(in (1.12)  $\{y, f\}$ ,  $\{z, g\} \in \mathcal{T}_{max}$  and  $P_2$  is the orthoprojector in  $\mathcal{H}_0$  onto  $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$ ). Finally, a decomposing boundary triplet for  $T_{max}$  is defined as a collection  $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ , in which  $\Gamma_j : T_{max} \rightarrow \mathcal{H}_j$ ,  $j \in \{0, 1\}$ , are the linear maps given by

$$(1.13) \quad \Gamma_0\{\tilde{y}, \tilde{f}\} = \Gamma'_0 y, \quad \Gamma_1\{\tilde{y}, \tilde{f}\} = \Gamma'_1 y, \quad \{\tilde{y}, \tilde{f}\} \in T_{max}.$$

In the case of equal deficiency indices  $n_+ = n_-$  one has

$$\mathcal{H} = H_0 \oplus \mathcal{H}_b (:= \mathcal{H}_0 = \mathcal{H}_1)$$

and the decomposing boundary triplet takes the form  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ , where

$$(1.14) \quad \Gamma_0\{\tilde{y}, \tilde{f}\} = \{-\Gamma_{1a} y + i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)y, \Gamma_{0b} y\} (\in H_0 \oplus \mathcal{H}_b),$$

$$(1.15) \quad \Gamma_1\{\tilde{y}, \tilde{f}\} = \{\Gamma_{0a} y + \frac{1}{2}(\widehat{\Gamma}_a + \widehat{\Gamma}_b)y, -\Gamma_{1b} y\} (\in H_0 \oplus \mathcal{H}_b), \quad \{\tilde{y}, \tilde{f}\} \in T_{max}.$$

Moreover, for the Hamiltonian system with  $n_+ = n_-$  one has  $\mathcal{H} = H \oplus \mathcal{H}_b$  and

$$(1.16) \quad \Gamma_0\{\tilde{y}, \tilde{f}\} = \{-\Gamma_{1a} y, \Gamma_{0b} y\} (\in H \oplus \mathcal{H}_b), \quad \Gamma_1\{\tilde{y}, \tilde{f}\} = \{\Gamma_{0a} y, -\Gamma_{1b} y\} (\in H \oplus \mathcal{H}_b).$$

It turns out that  $\Gamma_b y$  can be represented as a singular boundary value  $y_b$  of a function  $y \in \text{dom } \mathcal{T}_{max}$  (for more details see Remark 3.3). Therefore the operators (1.14) and (1.15) are defined, in fact, by means of boundary values of a function  $y$  at the endpoints  $a$  (regular value) and  $b$  (singular value). At the same time emphasize that a concrete form of the map  $\Gamma_b$  satisfying (1.11) does not matter, which is suitable for a compact representation of boundary conditions. To illustrate this assertion note that according to [38] self-adjoint separated boundary conditions exists only for a Hamiltonian system (1.2) with  $n_+ = n_-$ , in which case the general form of such conditions is

$$(1.17) \quad \cos B_1 y_0(a) + \sin B_1 y_1(a) = 0,$$

$$(1.18) \quad \cos B_2 \Gamma_{0b} y + \sin B_2 \Gamma_{1b} y = 0, \quad y \in \text{dom } \mathcal{T}_{max},$$

with self-adjoint operators  $B_1 \in [H]$  and  $B_2 \in [\mathcal{H}_b]$ . Formulas (1.17) and (1.18) seem to be more convenient than (1.7), because they enable one to parametrize regular self-adjoint boundary conditions (1.17) (at the point  $a$ ) and singular ones (1.18) (at the point  $b$ ) by means of self-adjoint boundary parameters  $B_1$  and  $B_2$  respectively.

In the present paper we investigate boundary value problems for general (not necessarily Hamiltonian) symmetric systems (1.2) with the aid of decomposing boundary triplets. We do not impose any restrictions on the deficiency indices  $n_{\pm}$  of  $T_{min}$ . To cover the case  $n_+ \neq n_-$  we consider the following problems:

- to find and describe  $\lambda$ -depending Nevanlinna (in particular, self-adjoint) boundary conditions which are analogs of self-adjoint separated boundary conditions for Hamiltonian systems;

- to find the operator functions which are analogs of the Titchmarsh-Weyl coefficient for Hamiltonian systems and describe these functions in terms of boundary conditions.

We suppose that solution of these problems will give rise to generalized Fourier transforms for the system (1.2) with the spectral functions of the minimally possible dimension. Our investigations are based on a fact that a decomposing boundary triplet  $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  is a boundary triplet for  $T_{max}$  in the sense of [33]; moreover, in the case  $n_+ = n_-$  a decomposing triplet  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  is a boundary triplet (boundary value space) for  $T_{max}$  in the sense of [16, 30]. This makes it possible to apply to the systems (1.2) the general theory of boundary triplets for abstract symmetric relations in Hilbert spaces (see [16, 5, 4, 30, 33] and references therein).

Assume for simplicity that  $n_+ = n_-$  and let  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (1.14), (1.15) for  $T_{max}$ . By using the results in [4, 30] we show that

$$(1.19) \quad T := \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = \hat{\Gamma}_by, \Gamma_{0b}y = \Gamma_{1b}y = 0\}$$

is a symmetric extension of  $T_{min}$  and each generalized resolvent  $R(\lambda)$  of  $T$  is defined by  $R(\lambda)\tilde{f} = \pi(y_f(\cdot, \lambda))$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , where  $f \in \mathcal{L}_{\Delta}^2(\mathcal{I})$ ,  $\pi f = \tilde{f}$  and  $y_f(\cdot, \lambda)$  is the  $\mathcal{L}_{\Delta}^2$ -solution of the following boundary value problem:

$$(1.20) \quad Jy' - B(t)y = \lambda\Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},$$

$$(1.21) \quad \Gamma_{1a}y = 0, \quad \hat{\Gamma}_ay = \hat{\Gamma}_by,$$

$$(1.22) \quad C_0(\lambda)\Gamma_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Here  $C_0(\lambda) \in [\mathcal{H}_b]$  and  $C_1(\lambda) \in [\mathcal{H}_b]$  are components of a Nevanlinna operator pair  $\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda))\}$ , so that (1.22) defines a Nevanlinna boundary condition at the singular endpoint  $b$ . A pair  $\tau = \tau(\lambda)$  plays a role of a boundary parameter, since  $R(\lambda)$  runs over the set of generalized resolvents of  $T$  when  $\tau(\lambda)$  runs over the set  $\tilde{R}(\mathcal{H}_b)$  of all Nevanlinna operator pairs. To emphasize this fact we write  $R(\lambda) = R_{\tau}(\lambda)$ . Observe also that a particular case of a boundary parameter  $\tau \in \tilde{R}(\mathcal{H}_b)$  is  $\tau(\lambda) = \{(I, K(\lambda))\}$ , where  $K(\lambda)$  is a Nevanlinna operator function.

The boundary problem (1.20)-(1.22) defines a canonical resolvent  $R_{\tau}(\lambda)$  if and only if  $\tau$  is a self-adjoint operator pair  $\tau = \{(\cos B, \sin B)\}$  with some  $B = B^* \in [\mathcal{H}_b]$ . In this case

$$(1.23) \quad R_{\tau}(\lambda) = (\tilde{A}_{\tau} - \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where  $\tilde{A}_{\tau}$  is a self-adjoint extension of  $T_{min}$  defined by the following mixed boundary conditions (c.f. (1.17) and (1.18)):

$$\Gamma_{1a}y = 0, \quad \hat{\Gamma}_ay = \hat{\Gamma}_by, \quad \cos B \cdot \Gamma_{0b}y + \sin B \cdot \Gamma_{1b}y = 0.$$

For each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  denote by  $\widehat{\mathfrak{N}}_\lambda(\subset T_{max})$  the subspace of all  $\{\widetilde{y}, \widetilde{f}\} \in T_{max}$  such that  $\widetilde{f} = \lambda \widetilde{y}$ . According to [5, 30] one associates with the decomposing boundary triplet  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  for  $T_{max}$  the  $\gamma$ -field  $\gamma(\lambda) \in [\mathcal{H}, L^2_\Delta(\mathcal{I})]$  and the abstract Weyl function  $M(\lambda) \in [\mathcal{H}]$  defined by

$$(1.24) \quad \gamma(\lambda) = \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda)^{-1}, \quad M(\lambda)h = \Gamma_1\{\gamma(\lambda)h, \lambda\gamma(\lambda)h\}, \quad h \in \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

It turns out that the  $\gamma$ -field satisfies the equality

$$(\gamma(\lambda)h)(t) = \pi(Z(t, \lambda)h), \quad h \in \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

with some operator  $\mathcal{L}^2_\Delta$ -solution  $Z(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}, \mathbb{H}]$  of Eq. (1.4). This fact enables us to show, that for each Nevanlinna boundary parameter  $\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda))\}$  there exists a unique operator  $\mathcal{L}^2_\Delta$ -solution  $v_\tau(\cdot, \lambda) \in \mathcal{L}^2_\Delta[H_0, \mathbb{H}]$  ( $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ) of Eq. (1.4) satisfying the boundary conditions

$$(1.25) \quad \begin{aligned} \Gamma_{1a}(v_\tau(t, \lambda)h_0) &= -P_H h_0, \\ i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)(v_\tau(t, \lambda)h_0) &= P_{\widehat{H}} h_0, \end{aligned}$$

$$(1.26) \quad C_0(\lambda)\Gamma_{0b}(v_\tau(t, \lambda)h_0) + C_1(\lambda)\Gamma_{1b}(v_\tau(t, \lambda)h_0) = 0, \quad h_0 \in H_0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

(here  $P_H$  and  $P_{\widehat{H}}$  are the orthoprojectors in  $H_0$  onto  $H$  and  $\widehat{H}$  respectively). By using the solution  $v_\tau(\cdot, \lambda)$  we introduce the concept of the  $m$ -function  $m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [H_0]$  corresponding to the boundary parameter  $\tau$  or, equivalently, to the boundary value problem (1.20)-(1.22). This function is defined by the following statement:

— for each  $\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda))\} \in \widetilde{R}(\mathcal{H}_b)$  there exists a unique operator function  $m_\tau(\lambda) \in [H_0]$  such that the operator solution

$$(1.27) \quad v_\tau(t, \lambda) := \varphi(t, \lambda)m_\tau(\lambda) + \psi(t, \lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

of Eq. (1.4) belongs to  $\mathcal{L}^2_\Delta[H_0, \mathbb{H}]$  and satisfies the boundary conditions (1.25) and (1.26).

Here  $\varphi(\cdot, \lambda)$  and  $\psi(\cdot, \lambda)$  are the  $[H_0, \mathbb{H}]$ -valued solutions of Eq. (1.4) with the initial data

$$(1.28) \quad X_a \varphi(a, \lambda) = \begin{pmatrix} I_{H_0} \\ 0 \end{pmatrix} \in [H_0, H_0 \oplus H], \quad X_a \psi(a, \lambda) = \begin{pmatrix} -\frac{i}{2}P_{\widehat{H}} \\ -P_H \end{pmatrix} \in [H_0, H_0 \oplus H].$$

The  $m$ -function  $m_\tau(\cdot)$  is called canonical if  $\tau = \{(\cos B, \sin B)\}$  is a self-adjoint operator pair or, equivalently, if  $m_\tau(\cdot)$  corresponds to the canonical resolvent (1.23). In this case the boundary condition (1.26) can be written as

$$(1.29) \quad \cos B \cdot \Gamma_{0b}(v_\tau(t, \lambda)h_0) + \sin B \cdot \Gamma_{1b}(v_\tau(t, \lambda)h_0) = 0, \quad h_0 \in H_0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

It turns out that under the special choice of the maps  $\Gamma_{0b}$  and  $\Gamma_{1b}$  the condition (1.29) takes the form of the second relation in (1.9). This and (1.27) imply that in the case of the Hamiltonian system (1.2) the canonical  $m$ -function  $m_\tau(\cdot)$  coincides with the Titchmarsh-Weyl coefficient  $M_{TW}(\cdot)$  in the sense of [19] (for more details see Remark 6.11).

We show in the paper that all  $m$ -functions can be parametrized immediately in terms of the Nevanlinna boundary parameter  $\tau$  by means of the formula similar to the known Krein formula for resolvents. More precisely the following theorem holds

**Theorem 1.1.** *Let  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet for  $T_{max}$  and let*

$$(1.30) \quad M(\lambda) = \begin{pmatrix} m_0(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : H_0 \oplus \mathcal{H}_b \rightarrow H_0 \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

be the block representation of the Weyl function (1.24). Then for every Nevanlinna boundary parameter  $\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda))\}$  the corresponding  $m$ -function  $m_\tau(\cdot)$  is of the form

$$(1.31) \quad m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Note that a description of all canonical  $m$ -functions of a differential operator in the case of maximal deficiency indices of the minimal operator can be found in [15, 13, 21]; similar result for Hamiltonian systems was obtained in [18]. In these papers each canonical  $m$ -function  $m_\tau(\cdot)$  is represented as a certain linear fractional transformation of a self-adjoint boundary parameter  $\tau$ . Observe also that for a differential operator of an even order with arbitrary (possibly unequal) deficiency indices a description of  $m$ -functions in the form (1.31) was obtained in [34].

It turns out that  $m_\tau(\cdot)$  is a Nevanlinna operator function satisfying the inequality

$$(1.32) \quad (\operatorname{Im} \lambda)^{-1} \cdot \operatorname{Im} m_\tau(\lambda) \geq \int_{\mathcal{I}} v_\tau^*(t, \lambda) \Delta(t) v_\tau(t, \lambda) dt, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

Moreover, the canonical  $m$ -function  $m_\tau(\cdot)$  satisfies the identity

$$(1.33) \quad m_\tau(\mu) - m_\tau^*(\lambda) = (\mu - \bar{\lambda}) \int_{\mathcal{I}} v_\tau^*(t, \lambda) \Delta(t) v_\tau(t, \mu) dt, \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R},$$

which implies that for the canonical  $m$ -function the inequality (1.32) turns into the equality. The identity (1.33) follows from the fact that  $m_\tau(\cdot)$  is the abstract Weyl function of a boundary triplet for some symmetric extension of  $T_{\min}$ . Note that for the Titchmarsh-Weyl coefficient  $M_{TW}(\cdot)$  of the Hamiltonian system the identity (1.33) was proved in [19].

In the case of minimal equal deficiency indices  $n_+ = n_- (= \nu_-)$  the extension  $T$  in (1.19) is self-adjoint and the boundary condition (1.22) vanishes. Therefore in this case there exists a unique (canonical)  $m$ -function  $m(\cdot)$  of the problem (1.20), (1.21), which coincides with the abstract Weyl function  $M(\lambda)$  (see (1.24)).

Actually we consider symmetric systems with arbitrary (possibly unequal) deficiency indices  $n_\pm$ . To this end we use the decomposing boundary triplet  $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  with possibly unequal Hilbert spaces  $\mathcal{H}_0$  and  $\mathcal{H}_1$  (see (1.13)), which enables us to obtain the results similar to those specified above for the case  $n_+ = n_-$ . In particular, we define the  $m$ -function  $m_\tau(\lambda) (\in [H_0])$  and describe all the  $m$ -functions by means of formulas similar to (1.31). It turns that  $m_\tau(\cdot)$  is a Nevanlinna function, which in the case  $n_+ < n_-$  has the triangular form

$$(1.34) \quad m_\tau(\lambda) = \begin{pmatrix} m_{1,\tau}(\lambda) & 0 \\ m_{+,\tau}(\lambda) & \frac{i}{2}I \end{pmatrix}, \quad \lambda \in \mathbb{C}_+.$$

Emphasize that for the system (1.2) with  $n_+ \neq n_-$  there are no longer canonical  $m$ -functions.

The simplest situation is in the case of minimal deficiency indices  $n_\pm = \nu_\pm$  (for not Hamiltonian systems (1.2) this implies that  $n_+ < n_-$ ). In this case there exists a unique  $m$ -function  $m(\cdot)$ , which has the triangular form

$$(1.35) \quad m(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ N_+(\lambda) & \frac{i}{2}I_{\widehat{H}} \end{pmatrix} : H \oplus \widehat{H} \rightarrow H \oplus \widehat{H}, \quad \lambda \in \mathbb{C}_+.$$

Here the entries  $M(\lambda)$  and  $N_+(\lambda)$  are taken from the block representation

$$(1.36) \quad M_+(\lambda) = (M(\lambda) : N_+(\lambda))^\top : H \rightarrow H \oplus \widehat{H}, \quad \lambda \in \mathbb{C}_+,$$

of the abstract Weyl function  $M_+(\cdot)$  corresponding to the decomposing boundary triplet  $\Pi$  (see Definition 2.11). Note in this connection that the systems (1.2) with minimal deficiency

indices  $n_{\pm}$  were studied in the paper by Hinton and Schneider [20], where the concept of the "rectangular" Titchmarsh-Weyl coefficient  $M_{TW}(\lambda) (\in [H, H \oplus \widehat{H}])$ ,  $\lambda \in \mathbb{C}_+$ , was introduced. This coefficient is defined by the relation

$$(1.37) \quad \varphi(t, \lambda) M_{TW}(\lambda) + \chi(t, \lambda) \in \mathcal{L}_{\Delta}^2[H, \mathbb{H}], \quad \lambda \in \mathbb{C}_+,$$

where  $\varphi(t, \lambda) (\in [H_0, \mathbb{H}])$  and  $\chi(t, \lambda) (\in [H, \mathbb{H}])$  are the operator solutions of Eq. (1.4) with the initial data

$$(1.38) \quad X_a \varphi(a, \lambda) = \begin{pmatrix} I_{H_0} \\ 0 \end{pmatrix} (\in [H_0, H_0 \oplus H]), \quad X_a \chi(a, \lambda) = \begin{pmatrix} 0 \\ -I_H \end{pmatrix} (\in [H, H_0 \oplus H])$$

(c.f. (1.28)). It is not difficult to prove that the abstract Weyl function (1.36) coincides with  $M_{TW}(\lambda)$  (see Remark 6.3).

In the final part of the paper we consider the operators generated by a differential expression  $l[y]$  of an odd order  $r = 2n + 1$  defined on an interval  $\mathcal{I} = [a, b)$  (see (7.1)). Such differential operators have been investigated in the papers by Everitt and Krishna Kumar [10, 11, 12, 27], where the limiting process from the compact intervals  $[a, \beta] \subset \mathcal{I}$  was used for construction of  $(n + 1)$ -component operator  $L^2$ -solutions  $v(t, \lambda)$  of the equation  $l[y] = \lambda y$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . With each solution  $v(t, \lambda)$  the authors associate certain boundary conditions and the Titchmarsh-Weyl matrix  $M_{TW}(\lambda) = (m_{rs}(\lambda))_{r,s=1}^{k+1}$ . These results are not completed; in particular, they do not enable to define self-adjoint boundary conditions without some hardly verifiable assumptions even in the case of equal minimally possible deficiency indices  $n_+(L_0) = n_-(L_0) = n + 1$  of the minimal operator  $L_0$ .

Our approach is based on the known fact [24] that the equation  $l[y] = \lambda y$  is equivalent to some symmetric not Hamiltonian system (1.4). This enables us to extend the results obtained for symmetric systems to differential operators of an odd order with arbitrary deficiency indices  $n_{\pm}(L_0)$ . In particular, we define the  $m$ -function  $m_{\tau}(\cdot)$  of such an operator and describe all  $m$ -functions immediately in terms of a Nevanlinna boundary parameter  $\tau$ .

Note in conclusion that the Green's functions of generalized resolvents  $R_{\tau}(\lambda)$  and the generalized Fourier transform for symmetric systems will be considered in the forthcoming paper.

## 2. PRELIMINARIES

**2.1. Notations.** The following notations will be used throughout the paper:  $\mathfrak{H}$ ,  $\mathcal{H}$  denote Hilbert spaces;  $[\mathcal{H}_1, \mathcal{H}_2]$  is the set of all bounded linear operators defined on the Hilbert space  $\mathcal{H}_1$  with values in the Hilbert space  $\mathcal{H}_2$ ;  $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$ ;  $A \upharpoonright \mathcal{L}$  is the restriction of an operator  $A$  onto the linear manifold  $\mathcal{L}$ ;  $P_{\mathcal{L}}$  is the orthogonal projector in  $\mathfrak{H}$  onto the subspace  $\mathcal{L} \subset \mathfrak{H}$ ;  $\mathbb{C}_+$  ( $\mathbb{C}_-$ ) is the upper (lower) half-plane of the complex plane.

Recall that a closed linear relation from  $\mathcal{H}_0$  to  $\mathcal{H}_1$  is a closed linear subspace in  $\mathcal{H}_0 \oplus \mathcal{H}_1$ . The set of all closed linear relations from  $\mathcal{H}_0$  to  $\mathcal{H}_1$  (in  $\mathcal{H}$ ) will be denoted by  $\tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$  ( $\tilde{\mathcal{C}}(\mathcal{H})$ ). A closed linear operator  $T$  from  $\mathcal{H}_0$  to  $\mathcal{H}_1$  is identified with its graph  $\text{gr } T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ .

For a linear relation  $T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$  we denote by  $\text{dom } T$ ,  $\text{ran } T$ ,  $\text{ker } T$  and  $\text{mul } T$  the domain, range, kernel and the multivalued part of  $T$  respectively. Recall also that the inverse and adjoint linear relations of  $T$  are the relations  $T^{-1} \in \tilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$  and  $T^* \in \tilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$

defined by

$$(2.1) \quad \begin{aligned} T^{-1} &= \{\{h_1, h_0\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : \{h_0, h_1\} \in T\} \\ T^* &= \{\{k_1, k_0\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : (k_0, h_0) - (k_1, h_1) = 0, \{h_0, h_1\} \in T\}. \end{aligned}$$

In the case  $T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$  we write  $0 \in \rho(T)$  if  $\ker T = \{0\}$  and  $\text{ran } T = \mathcal{H}_1$ , or equivalently if  $T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$ ;  $0 \in \hat{\rho}(T)$  if  $\ker T = \{0\}$  and  $\text{ran } T$  is a closed subspace in  $\mathcal{H}_1$ . For a linear relation  $T \in \tilde{\mathcal{C}}(\mathcal{H})$  we denote by  $\rho(T) := \{\lambda \in \mathbb{C} : 0 \in \rho(T - \lambda)\}$  and  $\hat{\rho}(T) = \{\lambda \in \mathbb{C} : 0 \in \hat{\rho}(T - \lambda)\}$  the resolvent set and the set of regular type points of  $T$  respectively.

Recall also the following definition.

**Definition 2.1.** A holomorphic operator function  $\Phi(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\mathcal{H}]$  is called a Nevanlinna function if  $\text{Im } \lambda \cdot \text{Im } \Phi(\lambda) \geq 0$  and  $\Phi^*(\lambda) = \Phi(\bar{\lambda})$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

**2.2. Holomorphic operator pairs.** Let  $\Lambda$  be an open set in  $\mathbb{C}$ , let  $\mathcal{K}, \mathcal{H}_0, \mathcal{H}_1$  be Hilbert spaces and let  $C_j(\cdot) : \Lambda \rightarrow [\mathcal{H}_j, \mathcal{K}]$ ,  $j \in \{0, 1\}$  be a pair of holomorphic operator functions (in short a holomorphic pair). Two such pairs  $C_j(\cdot) : \Lambda \rightarrow [\mathcal{H}_j, \mathcal{K}]$  and  $C'_j(\cdot) : \Lambda \rightarrow [\mathcal{H}_j, \mathcal{K}']$  are said to be equivalent if there exists a holomorphic isomorphism  $\varphi(\cdot) : \Lambda \rightarrow [\mathcal{K}, \mathcal{K}']$  such that  $C'_j(\lambda) = \varphi(\lambda)C_j(\lambda)$ ,  $\lambda \in \Lambda$ ,  $j \in \{0, 1\}$ . Clearly, the set of all holomorphic pairs splits into disjoint equivalence classes; moreover, the equality

$$(2.2) \quad \tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K}\} := \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0(\lambda)h_0 + C_1(\lambda)h_1 = 0\}$$

allows us to identify such a class with the  $\tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ -valued function  $\tau(\lambda)$ ,  $\lambda \in \Lambda$ .

In what follows, unless otherwise stated,  $\mathcal{H}_0$  is a Hilbert space,  $\mathcal{H}_1$  is a subspace in  $\mathcal{H}_0$ ,  $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$  and  $P_j$  is the orthoprojector in  $\mathcal{H}_0$  onto  $\mathcal{H}_j$ ,  $j \in \{1, 2\}$ .

Let  $\alpha \in \{-1, +1\}$ . With each linear relation  $\theta \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$  we associate the  $\times$ -adjoint linear relation  $\theta_\alpha^\times \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$  given by

$$\theta_\alpha^\times = \{\{k_0, k_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : (k_1, h_0) - (k_0, h_1) + i\alpha(P_2 k_0, P_2 h_0) = 0 \text{ for all } \{h_0, h_1\} \in \theta\}.$$

It follows from (2.1) that in the case  $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$  one has  $\theta_\alpha^\times = \theta^*$ .

Next assume that  $\mathcal{K}_+$  and  $\mathcal{K}_-$  are auxiliary Hilbert spaces and

$$(2.3) \quad \tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K}_+\}, \quad \lambda \in \mathbb{C}_+; \quad \tau_-(\lambda) = \{(D_0(\lambda), D_1(\lambda)); \mathcal{K}_-\}, \quad \lambda \in \mathbb{C}_-$$

are equivalence classes of the holomorphic pairs

$$(2.4) \quad (C_0(\lambda) : C_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{K}_+, \quad \lambda \in \mathbb{C}_+$$

$$(2.5) \quad (D_0(\lambda) : D_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \rightarrow \mathcal{K}_-, \quad \lambda \in \mathbb{C}_-.$$

Assume also that

$$C_0(\lambda) = (C_{01}(\lambda) : C_{02}(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{K}_+; \quad D_0(\lambda) = (D_{01}(\lambda) : D_{02}(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{K}_-$$

are the block representations of  $C_0(\lambda)$  and  $D_0(\lambda)$ .

**Definition 2.2.** Let as before  $\alpha \in \{-1, +1\}$ . A collection  $\tau = \{\tau_+, \tau_-\}$  of two holomorphic pairs (2.3) (more precisely, of the equivalence classes of the corresponding pairs) belongs to



the class  $\tilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$  if it satisfies the following relations:

$$(2.6) \quad 2 \operatorname{Im}(C_1(\lambda)C_{01}^*(\lambda)) + \alpha C_{02}(\lambda)C_{02}^*(\lambda) \geq 0, \quad \lambda \in \mathbb{C}_+$$

$$(2.7) \quad 2 \operatorname{Im}(D_1(\lambda)D_{01}^*(\lambda)) + \alpha D_{02}(\lambda)D_{02}^*(\lambda) \leq 0, \quad \lambda \in \mathbb{C}_-$$

$$(2.8) \quad C_1(\lambda)D_{01}^*(\bar{\lambda}) - C_{01}(\lambda)D_1^*(\bar{\lambda}) + i\alpha C_{02}(\lambda)D_{02}^*(\bar{\lambda}) = 0, \quad \lambda \in \mathbb{C}_+$$

$$(2.9) \quad \text{if } \alpha = +1, \text{ then } 0 \in \rho(C_0(\lambda) - iC_1(\lambda)P_1) \text{ and } 0 \in \rho(D_{01}(\lambda) + iD_1(\lambda))$$

$$(2.10) \quad \text{if } \alpha = -1, \text{ then } 0 \in \rho(C_{01}(\lambda) - iC_1(\lambda)) \text{ and } 0 \in \rho(D_0(\lambda) + iD_1(\lambda)P_1).$$

A collection  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$  belongs to the class  $\tilde{R}_\alpha^0(\mathcal{H}_0, \mathcal{H}_1)$  if for some (and hence for any)  $\lambda \in \mathbb{C}_+$  one has

$$2 \operatorname{Im}(C_1(\lambda)C_{01}^*(\lambda)) + \alpha C_{02}(\lambda)C_{02}^*(\lambda) = 0,$$

$$0 \in \rho(C_{01}(\lambda) + iC_1(\lambda)) \text{ if } \alpha = +1 \text{ and } 0 \in \rho(C_0(\lambda) + iC_1(\lambda)P_1) \text{ if } \alpha = -1.$$

The following proposition is immediate from Definition 2.2 and the results of [32].

**Proposition 2.3.** 1) If  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$ , then  $(-\tau_\pm(\bar{\lambda}))_\alpha^\times = -\tau_\mp(\lambda)$ ,  $\lambda \in \mathbb{C}_\mp$ , and the following equality holds

$$(2.11) \quad \tau_\mp(\lambda) = \{ \{-h_1 - i\alpha P_2 h_0, -P_1 h_0\} : \{h_1, h_0\} \in (\tau_\pm(\bar{\lambda}))^* \}.$$

2) Each collection  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$  given by (2.3) satisfies the relations

$$(2.12) \quad \text{if } \alpha = +1, \text{ then } \dim \mathcal{K}_+ = \dim \mathcal{H}_0 \text{ and } \dim \mathcal{K}_- = \dim \mathcal{H}_1;$$

$$(2.13) \quad \text{if } \alpha = -1, \text{ then } \dim \mathcal{K}_+ = \dim \mathcal{H}_1 \text{ and } \dim \mathcal{K}_- = \dim \mathcal{H}_0$$

3) The set  $\tilde{R}_\alpha^0(\mathcal{H}_0, \mathcal{H}_1)$  is not empty if and only if  $\dim \mathcal{H}_0 = \dim \mathcal{H}_1$ . This implies that in the case  $\dim \mathcal{H}_0 < \infty$  the set  $\tilde{R}_\alpha^0(\mathcal{H}_0, \mathcal{H}_1)$  is not empty if and only if  $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ .

4) Each collection  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_\alpha^0(\mathcal{H}_0, \mathcal{H}_1)$  can be represented as a constant

$$(2.14) \quad \tau_\pm(\lambda) \equiv \{(C_0, C_1); \mathcal{K}\} = \theta(\in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)), \quad \lambda \in \mathbb{C}_\pm,$$

where  $C_j \in [\mathcal{H}_j, \mathcal{K}]$ ,  $j \in \{0, 1\}$  and  $(-\theta)_\alpha^\times = -\theta$ .

Moreover, one can easily prove the following proposition.

**Proposition 2.4.** If  $\dim \mathcal{H}_0 < \infty$ , then a collection  $\tau = \{\tau_+, \tau_-\}$  of two holomorphic pairs (2.3) belongs to the class  $\tilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$  if and only if it satisfies (2.6)–(2.8), (2.12), (2.13) and the following relations

$$(2.15) \quad \operatorname{ran}(C_0(\lambda) : C_1(\lambda)) = \mathcal{K}_+, \quad \lambda \in \mathbb{C}_+; \quad \operatorname{ran}(D_0(\lambda) : D_1(\lambda)) = \mathcal{K}_-, \quad \lambda \in \mathbb{C}_-.$$

*Remark 2.5.* 1) It follows from Proposition 2.3, 2) that for each collection  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$  one can put in the representation (2.3)  $\mathcal{K}_+ = \mathcal{H}_0$ ,  $\mathcal{K}_- = \mathcal{H}_1$  in the case  $\alpha = +1$  and  $\mathcal{K}_+ = \mathcal{H}_1$ ,  $\mathcal{K}_- = \mathcal{H}_0$  in the case  $\alpha = -1$ .

2) If  $\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H}$ , then the class  $\tilde{R}(\mathcal{H}) := \tilde{R}_\alpha(\mathcal{H}, \mathcal{H})$  ( $\alpha \in \{-1, +1\}$ ) coincides with the well-known class of Nevanlinna functions  $\tau(\cdot)$  with values in  $\tilde{\mathcal{C}}(\mathcal{H})$  (see, for instance, [4]). In this case the collection (2.3) turns into the Nevanlinna pair

$$(2.16) \quad \tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{H}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

with  $C_0(\lambda), C_1(\lambda) \in [\mathcal{H}]$ . In view of (2.6)–(2.10) such a pair is characterized by the relations (cf. [4, Definition 2.2])

$$(2.17) \quad \operatorname{Im} \lambda \cdot \operatorname{Im}(C_1(\lambda)C_0^*(\lambda)) \geq 0, \quad C_1(\lambda)C_0^*(\bar{\lambda}) - C_0(\lambda)C_1^*(\bar{\lambda}) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(2.18) \quad 0 \in \rho(C_0(\lambda) - iC_1(\lambda)), \quad \lambda \in \mathbb{C}_+; \quad 0 \in \rho(C_0(\lambda) + iC_1(\lambda)), \quad \lambda \in \mathbb{C}_-.$$

Moreover, the function  $\tau(\cdot)$  belongs to the class  $\tilde{R}^0(\mathcal{H}) := \tilde{R}_\alpha^0(\mathcal{H}, \mathcal{H})$  if and only if it admits the representation in the form of the constant (cf. (2.14))

$$(2.19) \quad \tau(\lambda) \equiv \{(C_0, C_1); \mathcal{H}\} = \theta(\in \tilde{\mathcal{C}}(\mathcal{H})), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

with the operators  $C_j \in [\mathcal{H}]$  such that  $\operatorname{Im}(C_1C_0^*) = 0$  and  $0 \in \rho(C_0 \pm iC_1)$  (this means that  $\theta = \theta^*$ ). Observe also that according to [40] each  $\tau \in \tilde{R}^0(\mathcal{H})$  admits the normalized representation (2.19) with

$$(2.20) \quad C_0 = \cos B, \quad C_1 = \sin B, \quad B = B^* \in [\mathcal{H}].$$

Assume now that  $n := \dim \mathcal{H} < \infty$ ,  $e = \{e_j\}_1^n$  is an orthonormal basis in  $\mathcal{H}$ ,  $\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{H}\}$  is a pair of holomorphic operator-functions  $C_l(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\mathcal{H}]$  and  $C_l(\lambda) = (c_{kj,l}(\lambda))_{k,j=1}^n$  is the matrix representations of the operator  $C_l(\lambda)$ ,  $l \in \{0, 1\}$ , in the basis  $e$ . Then by Proposition 2.4  $\tau$  belongs to the class  $\tilde{R}(\mathcal{H})$  if and only if the matrices  $C_0(\lambda)$  and  $C_1(\lambda)$  satisfy (2.17) and the following equality:

$$\operatorname{rank}(C_0(\lambda) : C_1(\lambda)) = n, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, the operator pair  $\theta = \{(C_0, C_1); \mathcal{H}\}$  belongs to the class  $\tilde{R}^0(\mathcal{H})$  if and only if  $\operatorname{Im}(C_1C_0^*) = 0$  and  $\operatorname{rank}(C_0 : C_1) = n$  (here  $C_l = (c_{kj,l})_{k,j=1}^n$  is the matrix representation of the operator  $C_l$ ,  $l \in \{0, 1\}$ , in the basis  $e$ ). Note that such a "matrix" definition of the classes  $\tilde{R}(\mathcal{H})$  and  $\tilde{R}^0(\mathcal{H})$  in the case  $\dim \mathcal{H} < \infty$  can be found, e.g. in [8, 25]

**2.3. Boundary triplets and Weyl functions.** Let  $A$  be a closed symmetric linear relation in the Hilbert space  $\mathfrak{H}$ , let  $\mathfrak{N}_\lambda(A) = \ker(A^* - \lambda)$  ( $\lambda \in \tilde{\rho}(A)$ ) be a defect subspace of  $A$ , let  $\hat{\mathfrak{N}}_\lambda(A) = \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(A)\}$  and let  $n_\pm(A) := \dim \mathfrak{N}_\lambda(A) \leq \infty$ ,  $\lambda \in \mathbb{C}_\pm$  be deficiency indices of  $A$ . Denote by  $\operatorname{Ext}_A$  the set of all proper extensions of  $A$ , i.e., the set of all relations  $\tilde{A} \in \tilde{\mathcal{C}}(\mathfrak{H})$  such that  $A \subset \tilde{A} \subset A^*$ .

Next assume that  $\mathcal{H}_0$  is a Hilbert space,  $\mathcal{H}_1$  is a subspace in  $\mathcal{H}_0$  and  $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$ , so that  $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$ . Denote by  $P_j$  the orthoprojector in  $\mathcal{H}_0$  onto  $\mathcal{H}_j$ ,  $j \in \{1, 2\}$ .

**Definition 2.6.** Let  $\alpha \in \{-1, +1\}$ . A collection  $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ , where  $\Gamma_j : A^* \rightarrow \mathcal{H}_j$ ,  $j \in \{0, 1\}$  are linear mappings, is called a boundary triplet for  $A^*$ , if the mapping  $\Gamma : \hat{f} \rightarrow \{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\}$ ,  $\hat{f} \in A^*$ , from  $A^*$  into  $\mathcal{H}_0 \oplus \mathcal{H}_1$  is surjective and the following Green's identity

$$(2.21) \quad (f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\mathcal{H}_0} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{H}_0} + i\alpha(P_2 \Gamma_0 \hat{f}, P_2 \Gamma_0 \hat{g})_{\mathcal{H}_2}$$

holds for all  $\hat{f} = \{f, f'\}$ ,  $\hat{g} = \{g, g'\} \in A^*$ .

In the sequel we will also use the notation  $\Pi_+$  (resp.  $\Pi_-$ ) instead of  $\Pi_{+1}$  (resp.  $\Pi_{-1}$ ).

**Proposition 2.7.** Let  $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a boundary triplet for  $A^*$ . Then

$$(2.22) \quad \dim \mathcal{H}_1 = n_-(A) \leq n_+(A) = \dim \mathcal{H}_0, \quad \text{if } \alpha = +1;$$

$$(2.23) \quad \dim \mathcal{H}_1 = n_+(A) \leq n_-(A) = \dim \mathcal{H}_0, \quad \text{if } \alpha = -1.$$

Conversely for any symmetric relation  $A$  with  $n_-(A) \leq n_+(A)$  (resp.  $n_+(A) \leq n_-(A)$ ) there exists a boundary triplet  $\Pi_+$  (resp.  $\Pi_-$ ) for  $A^*$ .

**Proposition 2.8.** *Let  $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a boundary triplet for  $A^*$ . Then:*

- 1)  $\ker \Gamma_0 \cap \ker \Gamma_1 = A$  and  $\Gamma_j$  is a bounded operator from  $A^*$  into  $\mathcal{H}_j$ ,  $j \in \{0, 1\}$ ;
- 2) The equality

$$(2.24) \quad A_0 := \ker \Gamma_0 = \{\widehat{f} \in A^* : \Gamma_0 \widehat{f} = 0\}$$

defines the maximal symmetric extension  $A_0 \in \text{Ext}_A$  such that  $\mathbb{C}_+ \subset \rho(A_0)$  in the case  $\alpha = +1$  and  $\mathbb{C}_- \subset \rho(A_0)$  in the case  $\alpha = -1$ .

In the following two propositions we denote by  $\pi_1$  the orthoprojector in  $\mathfrak{H} \oplus \mathfrak{H}$  onto  $\mathfrak{H} \oplus \{0\}$ .

**Proposition 2.9.** *Let  $n_-(A) \leq n_+(A)$  and let  $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a boundary triplet for  $A^*$ . Then:*

- 1) the operators  $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A)$ ,  $\lambda \in \mathbb{C}_+$ , and  $P_1 \Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_z(A)$ ,  $z \in \mathbb{C}_-$ , isomorphically map  $\widehat{\mathfrak{N}}_\lambda(A)$  onto  $\mathcal{H}_0$  and  $\widehat{\mathfrak{N}}_z(A)$  onto  $\mathcal{H}_1$  respectively. Therefore the equalities

$$(2.25) \quad \gamma_+(\lambda) = \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A))^{-1}, \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(z) = \pi_1(P_1 \Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_z(A))^{-1}, \quad z \in \mathbb{C}_-,$$

$$(2.26) \quad M_+(\lambda)h_0 = \Gamma_1\{\gamma_+(\lambda)h_0, \lambda\gamma_+(\lambda)h_0\}, \quad h_0 \in \mathcal{H}_0, \quad \lambda \in \mathbb{C}_+$$

$$(2.27) \quad M_-(z)h_1 = (\Gamma_1 + iP_2\Gamma_0)\{\gamma_-(z)h_1, z\gamma_-(z)h_1\}, \quad h_1 \in \mathcal{H}_1, \quad z \in \mathbb{C}_-$$

correctly define the operator functions  $\gamma_+(\cdot) : \mathbb{C}_+ \rightarrow [\mathcal{H}_0, \mathfrak{H}]$ ,  $\gamma_-(\cdot) : \mathbb{C}_- \rightarrow [\mathcal{H}_1, \mathfrak{H}]$  and  $M_+(\cdot) : \mathbb{C}_+ \rightarrow [\mathcal{H}_0, \mathcal{H}_1]$ ,  $M_-(\cdot) : \mathbb{C}_- \rightarrow [\mathcal{H}_1, \mathcal{H}_0]$ , which are holomorphic on their domains. Moreover, the equality  $M_+^*(\overline{\lambda}) = M_-(\lambda)$ ,  $\lambda \in \mathbb{C}_-$ , is valid.

- 2) assume that

$$(2.28) \quad M_+(\lambda) = (M(\lambda) : N_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+$$

$$(2.29) \quad M_-(z) = (M(z) : N_-(z))^\top : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, \quad z \in \mathbb{C}_-$$

are the block representations of  $M_+(\lambda)$  and  $M_-(z)$  respectively and let

$$(2.30) \quad \mathcal{M}(\lambda) = \begin{pmatrix} M(\lambda) & N_+(\lambda) \\ 0 & \frac{i}{2}I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_+$$

$$(2.31) \quad \mathcal{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ N_-(\lambda) & -\frac{i}{2}I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_-.$$

Then  $\mathcal{M}(\cdot)$  is a Nevanlinna operator function satisfying the identity

$$(2.32) \quad \mathcal{M}(\mu) - \mathcal{M}^*(\lambda) = (\mu - \overline{\lambda})\gamma_+^*(\lambda)\gamma_+(\mu), \quad \mu, \lambda \in \mathbb{C}_+.$$

Similar statements for the triplet  $\Pi_-$  are specified in the following proposition.

**Proposition 2.10.** *Let  $n_+(A) \leq n_-(A)$  and let  $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a boundary triplet for  $A^*$ . Then:*

- 1) the equalities

$$(2.33) \quad \gamma_+(\lambda) = \pi_1(P_1 \Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A))^{-1}, \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(z) = \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_z(A))^{-1}, \quad z \in \mathbb{C}_-,$$

$$(2.34) \quad M_+(\lambda)h_1 = (\Gamma_1 - iP_2\Gamma_0)\{\gamma_+(\lambda)h_1, \lambda\gamma_+(\lambda)h_1\}, \quad h_1 \in \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+$$

$$(2.35) \quad M_-(z)h_0 = \Gamma_1\{\gamma_-(z)h_0, z\gamma_-(z)h_0\}, \quad h_0 \in \mathcal{H}_0, \quad z \in \mathbb{C}_-$$

correctly define the holomorphic operator functions  $\gamma_+(\cdot) : \mathbb{C}_+ \rightarrow [\mathcal{H}_1, \mathfrak{H}]$ ,  $\gamma_-(\cdot) : \mathbb{C}_- \rightarrow [\mathcal{H}_0, \mathfrak{H}]$  and  $M_+(\cdot) : \mathbb{C}_+ \rightarrow [\mathcal{H}_1, \mathcal{H}_0]$ ,  $M_-(\cdot) : \mathbb{C}_- \rightarrow [\mathcal{H}_0, \mathcal{H}_1]$ .

2) assume that

$$(2.36) \quad M_+(\lambda) = (M(\lambda) : N_+(\lambda))^\top : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_+$$

$$(2.37) \quad M_-(z) = (M(z) : N_-(z)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1, \quad z \in \mathbb{C}_-$$

are the block representations of  $M_+(\lambda)$  and  $M_-(z)$  respectively and let

$$(2.38) \quad \mathcal{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ N_+(\lambda) & \frac{i}{2}I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_+$$

$$(2.39) \quad \mathcal{M}(\lambda) = \begin{pmatrix} M(\lambda) & N_-(\lambda) \\ 0 & -\frac{i}{2}I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_-.$$

Then  $\mathcal{M}(\cdot)$  is a Nevanlinna operator function satisfying the identity

$$(2.40) \quad \mathcal{M}(\mu) - \mathcal{M}^*(\lambda) = (\mu - \bar{\lambda})\gamma_-^*(\lambda)\gamma_-(\mu), \quad \mu, \lambda \in \mathbb{C}_-.$$

**Definition 2.11.** The operator functions  $\gamma_\pm(\cdot)$  and  $M_\pm(\cdot)$  defined in Propositions 2.9 and 2.10 are called the  $\gamma$ -fields and the Weyl functions, respectively, corresponding to the boundary triplet  $\Pi_\alpha$ .

**Proposition 2.12.** Let  $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a boundary triplet for  $A^*$  and let  $\gamma_\pm(\cdot)$  and  $M_\pm(\cdot)$  be the corresponding  $\gamma$ -fields and Weyl functions respectively. Moreover, let the spaces  $\mathcal{H}_0$  and  $\mathcal{H}_1$  be decomposed as

$$\mathcal{H}_1 = \widehat{\mathcal{H}} \oplus \dot{\mathcal{H}}_1, \quad \mathcal{H}_0 = \widehat{\mathcal{H}} \oplus \dot{\mathcal{H}}_0$$

(so that  $\dot{\mathcal{H}}_0 = \dot{\mathcal{H}}_1 \oplus \mathcal{H}_2$ ) and let

$$\Gamma_0 = (\widehat{\Gamma}_0 : \dot{\Gamma}_0)^\top : A^* \rightarrow \widehat{\mathcal{H}} \oplus \dot{\mathcal{H}}_0, \quad \Gamma_1 = (\widehat{\Gamma}_1 : \dot{\Gamma}_1)^\top : A^* \rightarrow \widehat{\mathcal{H}} \oplus \dot{\mathcal{H}}_1$$

be the block representations of the operators  $\Gamma_0$  and  $\Gamma_1$ . Then:

1) The equality

$$\widetilde{A} = \{\widehat{f} \in A^* : \widehat{\Gamma}_0 \widehat{f} = \dot{\Gamma}_0 \widehat{f} = \dot{\Gamma}_1 \widehat{f} = 0\}$$

defines a closed symmetric extension  $\widetilde{A} \in \text{Ext}_A$  and the adjoint relation  $\widetilde{A}^*$  of  $\widetilde{A}$  is

$$\widetilde{A}^* = \{\widehat{f} \in A^* : \widehat{\Gamma}_0 \widehat{f} = 0\}.$$

If in addition  $n_\pm(A) < \infty$ , then the deficiency indices of  $\widetilde{A}$  are  $n_\pm(\widetilde{A}) = n_\pm(A) - \dim \widehat{\mathcal{H}}$ .

2) The collection  $\dot{\Pi}_\alpha = \{\dot{\mathcal{H}}_0 \oplus \dot{\mathcal{H}}_1, \dot{\Gamma}_0 \upharpoonright \widetilde{A}^*, \dot{\Gamma}_1 \upharpoonright \widetilde{A}^*\}$  is a boundary triplet for  $\widetilde{A}^*$ .

3) The  $\gamma$ -fields  $\dot{\gamma}_\pm(\cdot)$  and the Weyl functions  $\dot{M}_\pm(\cdot)$  corresponding to  $\dot{\Pi}_\alpha$  are given by

$$\dot{\gamma}_+(\lambda) = \gamma_+(\lambda) \upharpoonright \dot{\mathcal{H}}_0, \quad \dot{M}_+(\lambda) = P_{\dot{\mathcal{H}}_1} M_+(\lambda) \upharpoonright \dot{\mathcal{H}}_0, \quad \lambda \in \mathbb{C}_+$$

$$\dot{\gamma}_-(\lambda) = \gamma_-(\lambda) \upharpoonright \dot{\mathcal{H}}_1, \quad \dot{M}_-(\lambda) = P_{\dot{\mathcal{H}}_0} M_-(\lambda) \upharpoonright \dot{\mathcal{H}}_1, \quad \lambda \in \mathbb{C}_-$$

in the case  $\alpha = +1$  and by the same formulas with  $\dot{\mathcal{H}}_1$  ( $\dot{\mathcal{H}}_0$ ) in place of  $\dot{\mathcal{H}}_0$  (resp.  $\dot{\mathcal{H}}_1$ ) in the case  $\alpha = -1$ .

We omit the proof of Proposition 2.12, since it is similar to that of Proposition 4.1 in [4] (see also remark 2.16 below).

Recall further the following definition.

**Definition 2.13.** An operator function  $R(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\mathfrak{H}]$  is called a generalized resolvent of a symmetric linear relation  $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$  if there exist a Hilbert space  $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$  and a self-adjoint linear relation  $\widetilde{A} \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$  such that  $A \subset \widetilde{A}$  and  $R(\lambda) = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

$R(\cdot)$  is a canonical resolvent if and only if  $\widetilde{\mathfrak{H}} = \mathfrak{H}$ . In this case  $R(\lambda) = (\widetilde{A} - \lambda)^{-1}$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

**Theorem 2.14.** *Let  $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a boundary triplet for  $A^*$ . If  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$  is a collection of holomorphic pairs (2.3), then for every  $g \in \mathfrak{H}$  and  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  the abstract boundary value problem*

$$(2.41) \quad \{f, \lambda f + g\} \in A^*$$

$$(2.42) \quad C_0(\lambda)\Gamma_0\{f, \lambda f + g\} - C_1(\lambda)\Gamma_1\{f, \lambda f + g\} = 0, \quad \lambda \in \mathbb{C}_+$$

$$(2.43) \quad D_0(\lambda)\Gamma_0\{f, \lambda f + g\} - D_1(\lambda)\Gamma_1\{f, \lambda f + g\} = 0, \quad \lambda \in \mathbb{C}_-$$

has a unique solution  $f = f(g, \lambda)$  and the equality  $R(\lambda)g := f(g, \lambda)$  defines a generalized resolvent  $R(\lambda) = R_\tau(\lambda)$  of the relation  $A$ . Conversely, for each generalized resolvent  $R(\lambda)$  of  $A$  there exists a unique  $\tau \in \tilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$  such that  $R(\lambda) = R_\tau(\lambda)$ . Moreover,  $R_\tau(\lambda)$  is a canonical resolvent if and only if  $\tau \in \tilde{R}_\alpha^0(\mathcal{H}_0, \mathcal{H}_1)$ .

In the following corollary we reformulate the statement of Theorem 2.14 for parameters  $\tau$  of a special form.

**Corollary 2.15.** *Assume that  $\mathcal{H}'$  and  $\tilde{\mathcal{H}}_1$  are Hilbert spaces,  $\mathcal{H}_1$  is a subspace in  $\tilde{\mathcal{H}}_1$  and  $\Pi_- = \{(\mathcal{H}' \oplus \tilde{\mathcal{H}}_1) \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  is a boundary triplet for  $A^*$ . If  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_{-1}(\tilde{\mathcal{H}}_1, \mathcal{H}_1)$  is a collection (2.3), then the direct statement of Theorem 2.14 holds with the following boundary conditions in place of (2.42) and (2.43):*

$$C_0(\lambda)P_{\tilde{\mathcal{H}}_1}\Gamma_0\{f, \lambda f + g\} - C_1(\lambda)\Gamma_1\{f, \lambda f + g\} = 0, \quad \lambda \in \mathbb{C}_+$$

$$D_0(\lambda)P_{\tilde{\mathcal{H}}_1}\Gamma_0\{f, \lambda f + g\} - D_1(\lambda)\Gamma_1\{f, \lambda f + g\} = 0, \quad P_{\mathcal{H}'}\Gamma_0\{f, \lambda f + g\} = 0, \quad \lambda \in \mathbb{C}_-$$

*Proof.* Let  $\dot{\tau}_+(\lambda) = \{(\dot{C}_0(\lambda), \dot{C}_1(\lambda)), \mathcal{K}_+\}$ ,  $\lambda \in \mathbb{C}_+$ , and  $\dot{\tau}_-(\lambda) = \{(\dot{D}_0(\lambda), \dot{D}_1(\lambda)), \mathcal{H}' \oplus \mathcal{K}_-\}$ ,  $\lambda \in \mathbb{C}_-$ , be holomorphic operator pairs with

$$\dot{C}_0(\lambda) = C_0(\lambda)P_{\tilde{\mathcal{H}}_1} \in [\mathcal{H}' \oplus \tilde{\mathcal{H}}_1, \mathcal{K}_+], \quad \dot{C}_1(\lambda) = C_1(\lambda) \in [\mathcal{H}_1, \mathcal{K}_+]$$

$$\dot{D}_0(\lambda) = P_{\mathcal{H}'} + D_0(\lambda)P_{\tilde{\mathcal{H}}_1} \in [\mathcal{H}' \oplus \tilde{\mathcal{H}}_1, \mathcal{H}' \oplus \mathcal{K}_-], \quad \dot{D}_1(\lambda) = D_1(\lambda) \in [\mathcal{H}_1, \mathcal{H}' \oplus \mathcal{K}_-].$$

Then the direct calculations show that the operator functions  $\dot{C}_j(\cdot)$  and  $\dot{D}_j(\cdot)$ ,  $j \in \{0, 1\}$ , satisfy the relations (2.6)–(2.8), (2.10) and hence a collection  $\dot{\tau} = \{\dot{\tau}_+, \dot{\tau}_-\}$  belongs to  $\tilde{R}_{-1}(\mathcal{H}' \oplus \tilde{\mathcal{H}}_1, \mathcal{H}_1)$ . Applying now Theorem 2.14 to  $\dot{\tau}$  we arrive at the desired statement.  $\square$

*Remark 2.16.* 1) For  $\alpha = +1$  definition of the boundary triplet  $\Pi_\alpha = \Pi_+$  and the corresponding Weyl functions  $M_\pm(\cdot)$  are given in the paper [33]. Moreover, the proof of Propositions 2.7–2.9 and Theorem 2.14 for the triplets  $\Pi_+$  is adduced in this paper as well (for the triplets  $\Pi_-$  the proof is similar).

2) If  $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$ , then the triplet  $\Pi_\alpha$  turns into the boundary triplet (boundary value space)  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  for  $A^*$  in the sense of [16, 30]. In this case  $n_+(A) = n_-(A) = \dim \mathcal{H}$ ,  $A_0 (= \ker \Gamma_0)$  is a self-adjoint extension of  $A$  and according to [5, 30, 6] the relations

$$(2.44) \quad \gamma(\lambda) = \pi_1(\Gamma_0 \upharpoonright \hat{\mathfrak{N}}_\lambda(A))^{-1}, \quad \Gamma_1 \upharpoonright \hat{\mathfrak{N}}_\lambda(A) = M(\lambda)\Gamma_0 \upharpoonright \hat{\mathfrak{N}}_\lambda(A), \quad \lambda \in \rho(A_0)$$

define the  $\gamma$ -field  $\gamma(\cdot) : \rho(A_0) \rightarrow [\mathcal{H}, \mathfrak{H}]$  and the Weyl function  $M(\cdot) : \rho(A_0) \rightarrow [\mathcal{H}]$  corresponding to the triplet  $\Pi$ . It follows from (2.44) that  $\gamma(\cdot)$  and  $M(\cdot)$  are associated with the operator functions  $\gamma_\pm(\cdot)$  and  $M_\pm(\cdot)$  from Definition 2.11 via  $\gamma(\lambda) = \gamma_\pm(\lambda)$  and  $M(\lambda) = M_\pm(\lambda)$ ,  $\lambda \in \mathbb{C}_\pm$ . Moreover, for such a triplet the identity (2.32) takes the form

$$(2.45) \quad M(\mu) - M^*(\lambda) = (\mu - \bar{\lambda})\gamma^*(\lambda)\gamma(\mu), \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Observe also that for the triplet  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  all the results in this subsection were obtained in [5, 30, 6, 4].

In what follows a boundary triplet  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  in the sense of [16, 30] will be sometimes called an ordinary boundary triplet for  $A^*$ .

### 3. DECOMPOSING BOUNDARY TRIPLETS FOR SYMMETRIC SYSTEMS

**3.1. Notations.** Let  $\mathcal{I} = [a, b)$  ( $-\infty < a < b \leq \infty$ ) be an interval of the real line (in the case  $b < \infty$  the endpoint  $b$  may or may not belong to  $\mathcal{I}$ ), let  $\mathbb{H}$  be a finite-dimensional Hilbert space, let  $AC(\mathcal{I}; \mathbb{H})$  be the set of all functions  $f(\cdot) : \mathcal{I} \rightarrow \mathbb{H}$  which are absolutely continuous on each segment  $[a, \beta] \subset \mathcal{I}$  and let  $AC(\mathcal{I}) := AC(\mathcal{I}; \mathbb{C})$ . Denote also by  $\mathcal{L}_{loc}^1(\mathcal{I}; [\mathbb{H}])$  the set of all Borel operator functions  $F(\cdot)$  defined almost everywhere on  $\mathcal{I}$  with values in  $[\mathbb{H}]$  and such that  $\int_{[a, \beta]} \|F(t)\| dt < \infty$  for each  $\beta \in \mathcal{I}$ .

Next assume that  $\Delta(\cdot) \in \mathcal{L}_{loc}^1(\mathcal{I}; [\mathbb{H}])$  is an operator function such that  $\Delta(t) \geq 0$  a.e. on  $\mathcal{I}$  and let  $\mathcal{L}_{\Delta}^2(\mathcal{I})$  be the linear space of all Borel functions  $f(\cdot)$  defined almost everywhere on  $\mathcal{I}$  with values in  $\mathbb{H}$  and such that  $\int_{\mathcal{I}} (\Delta(t)f(t), f(t))_{\mathbb{H}} dt < \infty$ . Moreover, for a given finite-dimensional Hilbert space  $\mathcal{K}$  denote by  $\mathcal{L}_{\Delta}^2[\mathcal{K}, \mathbb{H}]$  the set of all Borel operator-functions  $F(\cdot) : \mathcal{I} \rightarrow [\mathcal{K}, \mathbb{H}]$  such that there exists the integral  $\int_{\mathcal{I}} F^*(t)\Delta(t)F(t) dt$ . It is clear that the latter condition is equivalent to  $F(t)h \in \mathcal{L}_{\Delta}^2(\mathcal{I})$  for each  $h \in \mathcal{K}$ .

As is known [22, 9]  $\mathcal{L}_{\Delta}^2(\mathcal{I})$  is a semi-Hilbert space with the semi-definite inner product  $(\cdot, \cdot)_{\Delta}$  and semi-norm  $\|\cdot\|_{\Delta}$  given by

$$(3.1) \quad (f, g)_{\Delta} = \int_{\mathcal{I}} (\Delta(t)f(t), g(t))_{\mathbb{H}} dt, \quad \|f\|_{\Delta} = ((f, f)_{\Delta})^{\frac{1}{2}}, \quad f, g \in \mathcal{L}_{\Delta}^2(\mathcal{I}).$$

The semi-Hilbert space  $\mathcal{L}_{\Delta}^2(\mathcal{I})$  gives rise to the Hilbert space  $L_{\Delta}^2(\mathcal{I}) = \mathcal{L}_{\Delta}^2(\mathcal{I}) / \{f \in \mathcal{L}_{\Delta}^2(\mathcal{I}) : \|f\|_{\Delta} = 0\}$ , i.e.,  $L_{\Delta}^2(\mathcal{I})$  is the Hilbert space of all equivalence classes. The inner product and norm in  $L_{\Delta}^2(\mathcal{I})$  are defined by

$$(\tilde{f}, \tilde{g}) = (f, g)_{\Delta}, \quad \|\tilde{f}\| = (\tilde{f}, \tilde{f})^{\frac{1}{2}} = \|f\|_{\Delta}, \quad \tilde{f}, \tilde{g} \in L_{\Delta}^2(\mathcal{I}),$$

where  $f \in \tilde{f}$  ( $g \in \tilde{g}$ ) is any representative of the class  $\tilde{f}$  (resp.  $\tilde{g}$ ).

In the sequel we systematically use the quotient map  $\pi$  from  $\mathcal{L}_{\Delta}^2(\mathcal{I})$  onto  $L_{\Delta}^2(\mathcal{I})$  given by  $\pi f = \tilde{f} (\ni f)$ ,  $f \in \mathcal{L}_{\Delta}^2(\mathcal{I})$ . Moreover, we let  $\tilde{\pi} = \pi \oplus \pi : (\mathcal{L}_{\Delta}^2(\mathcal{I}))^2 \rightarrow (L_{\Delta}^2(\mathcal{I}))^2$ , so that  $\tilde{\pi}\{f, g\} = \{\tilde{f}, \tilde{g}\}$ ,  $f, g \in \mathcal{L}_{\Delta}^2(\mathcal{I})$ .

**3.2. Symmetric systems.** In this subsection we provide some known results on symmetric systems of differential equations.

Let as above  $\mathcal{I} = [a, b)$  ( $-\infty < a < b \leq \infty$ ) be an interval and let  $\mathbb{H}$  be a Hilbert space with  $n := \dim \mathbb{H} < \infty$ . Moreover, let  $B(\cdot), \Delta(\cdot) \in \mathcal{L}_{loc}^1(\mathcal{I}; [\mathbb{H}])$  be operator functions such that  $B(t) = B^*(t)$  and  $\Delta(t) \geq 0$  a.e. on  $\mathcal{I}$  and let  $J \in [\mathbb{H}]$  be a signature operator (this means that  $J^* = J^{-1} = -J$ ).

A first-order symmetric system on an interval  $\mathcal{I}$  (with the regular endpoint  $a$ ) is a system of differential equations of the form

$$(3.2) \quad Jy'(t) - B(t)y(t) = \Delta(t)f(t), \quad t \in \mathcal{I},$$

where  $f(\cdot) \in \mathcal{L}_{\Delta}^2(\mathcal{I})$ . Together with (3.2) we consider also the homogeneous system

$$(3.3) \quad Jy'(t) - B(t)y(t) = \lambda \Delta(t)y(t), \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}.$$

A function  $y \in AC(\mathcal{I}; \mathbb{H})$  is a solution of (3.2) (resp. (3.3)) if the equality (3.2) (resp. (3.3)) holds a.e. on  $\mathcal{I}$ . Moreover, a function  $Y(\cdot, \lambda) : \mathcal{I} \rightarrow [\mathcal{K}, \mathbb{H}]$  is an operator solution of the

equation (3.3) if  $y(t) = Y(t, \lambda)h$  is a (vector) solution of this equation for each  $h \in \mathcal{K}$  (here  $\mathcal{K}$  is a Hilbert space with  $\dim \mathcal{K} < \infty$ ).

Everywhere below we suppose that the system (3.2) is definite in the sense of the following definition.

**Definition 3.1.** [14, 39, 24] The symmetric system (3.2) is called definite if for each  $\lambda \in \mathbb{C}$  and each solution  $y$  of (3.3) the equality  $\Delta(t)y(t) = 0$  (a.e. on  $\mathcal{I}$ ) implies  $y(t) = 0$ ,  $t \in \mathcal{I}$ .

As is known [39] the symmetric system (3.2) induces the *maximal relations*  $\mathcal{T}_{max}$  in  $\mathcal{L}_\Delta^2(\mathcal{I})$  and  $T_{max}$  in  $L_\Delta^2(\mathcal{I})$ , which are defined by

$$(3.4) \quad \mathcal{T}_{max} = \{\{y, f\} \in (\mathcal{L}_\Delta^2(\mathcal{I}))^2 : y \in AC(\mathcal{I}; \mathbb{H}) \text{ and } Jy'(t) - B(t)y(t) = \Delta(t)f(t) \text{ a.e. on } \mathcal{I}\}$$

and  $T_{max} = \tilde{\pi}\mathcal{T}_{max}$ . Moreover the Lagrange's identity

$$(3.5) \quad (f, z)_\Delta - (y, g)_\Delta = [y, z]_b - (Jy(a), z(a)), \quad \{y, f\}, \{z, g\} \in \mathcal{T}_{max}.$$

holds with

$$(3.6) \quad [y, z]_b := \lim_{t \uparrow b} (Jy(t), z(t)), \quad y, z \in \text{dom } \mathcal{T}_{max}.$$

Formula (3.6) defines the boundary bilinear form  $[\cdot, \cdot]_b$  on  $\text{dom } \mathcal{T}_{max}$ , which plays an essential role in our considerations. By using this form we define the *minimal relations*  $\mathcal{T}_{min}$  in  $\mathcal{L}_\Delta^2(\mathcal{I})$  and  $T_{min}$  in  $L_\Delta^2(\mathcal{I})$  via

$$(3.7) \quad \mathcal{T}_{min} = \{\{y, f\} \in \mathcal{T}_{max} : y(a) = 0 \text{ and } [y, z]_b = 0 \text{ for each } z \in \text{dom } \mathcal{T}_{max}\}.$$

and  $T_{min} = \tilde{\pi}\mathcal{T}_{min}$ . According to [39]  $T_{min}$  is a closed symmetric linear relation in  $L_\Delta^2(\mathcal{I})$  and  $T_{min}^* = T_{max}$ .

For each  $\lambda \in \mathbb{C}$  denote by  $\mathcal{N}_\lambda$  the linear space of all solutions of the homogeneous system (3.3) belonging to  $\mathcal{L}_\Delta^2(\mathcal{I})$ . Definition (3.4) of  $\mathcal{T}_{max}$  implies that

$$\mathcal{N}_\lambda = \ker(\mathcal{T}_{max} - \lambda) = \{y \in \mathcal{L}_\Delta^2(\mathcal{I}) : \{y, \lambda y\} \in \mathcal{T}_{max}\}, \quad \lambda \in \mathbb{C}.$$

and hence  $\mathcal{N}_\lambda \subset \text{dom } \mathcal{T}_{max}$ .

Assume that

$$n_\pm := n_\pm(T_{min}) = \dim \mathfrak{N}_\lambda(T_{min}), \quad \lambda \in \mathbb{C}_\pm,$$

are deficiency indices of  $T_{min}$ . It is easily seen that  $\pi\mathcal{N}_\lambda = \mathfrak{N}_\lambda(T_{min})$  and  $\ker(\pi \upharpoonright \mathcal{N}_\lambda) = \{0\}$ ,  $\lambda \in \mathbb{C}$ . This implies that  $\dim \mathcal{N}_\lambda = n_\pm$ ,  $\lambda \in \mathbb{C}_\pm$ .

Let  $J \in [\mathbb{H}]$  be the signature operator in (3.2) and let

$$\nu_+ = \dim \ker(iJ - I) \quad \text{and} \quad \nu_- = \dim \ker(iJ + I).$$

In what follows we suppose that

$$(3.8) \quad \hat{\nu} := \nu_- - \nu_+ \geq 0.$$

In this case one can assume without loss of generality that the following statements hold:

(i) the Hilbert space  $\mathbb{H}$  is of the form

$$(3.9) \quad \mathbb{H} = H \oplus \hat{H} \oplus H,$$

where  $H$  and  $\hat{H}$  are finite dimensional Hilbert spaces with

$$(3.10) \quad \dim H = \nu_+, \quad \dim \hat{H} = \hat{\nu};$$

(ii) the operator  $J$  is of the form (1.3).

Introducing the Hilbert space

$$(3.11) \quad H_0 = H \oplus \widehat{H}$$

one can represent the equality (3.9) as

$$(3.12) \quad \mathbb{H} = (H \oplus \widehat{H}) \oplus H = H_0 \oplus H.$$

Let  $\nu_{b+}$  and  $\nu_{b-}$  be indices of inertia of the skew-Hermitian bilinear form (3.6). Then  $\nu_{b\pm} < \infty$  and the following equality holds [2, 38]

$$(3.13) \quad n_+ = \nu_+ + \nu_{b+}, \quad n_- = \nu_- + \nu_{b-}.$$

Therefore  $T_{min}$  has equal deficiency indices  $n_+ = n_-$  if and only if

$$(3.14) \quad \widehat{\nu} = \nu_{b+} - \nu_{b-}.$$

Observe also that according to [38, Lemma 5.1] there exist Hilbert spaces  $\mathcal{H}_b$  and  $\widehat{\mathcal{H}}_b$  and a surjective linear map

$$(3.15) \quad \Gamma_b = (\Gamma_{0b} : \widehat{\Gamma}_b : \Gamma_{1b})^\top : \text{dom } \mathcal{T}_{max} \rightarrow \mathcal{H}_b \oplus \widehat{\mathcal{H}}_b \oplus \mathcal{H}_b$$

such that for all  $y, z \in \text{dom } \mathcal{T}_{max}$  the following equality is valid

$$(3.16) \quad [y, z]_b = i \cdot \text{sign}(\nu_{b+} - \nu_{b-}) (\widehat{\Gamma}_b y, \widehat{\Gamma}_b z) - (\Gamma_{1b} y, \Gamma_{0b} z) + (\Gamma_{0b} y, \Gamma_{1b} z).$$

Moreover, for such a map  $\Gamma_b$  one has  $\ker \Gamma_b = \ker [\cdot, \cdot]_b$  and

$$(3.17) \quad \dim \mathcal{H}_b = \min\{\nu_{b+}, \nu_{b-}\}, \quad \dim \widehat{\mathcal{H}}_b = |\nu_{b+} - \nu_{b-}|.$$

Recall that the system (3.2) is called regular if  $\mathcal{I} = [a, b]$  is a compact interval and both the integrals  $\int_{\mathcal{I}} \|B(t)\| dt$  and  $\int_{\mathcal{I}} \|\Delta(t)\| dt$  are finite. For a regular system one can put  $\mathcal{H}_b = H$ ,  $\widehat{\mathcal{H}}_b = \widehat{H}$  and  $\Gamma_b y = X_b y(b)$ ,  $y \in \text{dom } \mathcal{T}_{max}$ , where  $X_b \in [\mathbb{H}]$  and  $X_b^* J X_b = J$ .

Next assume that  $X_a \in [\mathbb{H}]$  is the operator such that  $X_a^* J X_a = J$  and let  $\Gamma_a : AC(\mathcal{I}; \mathbb{H}) \rightarrow \mathbb{H}$  be the linear map given by

$$(3.18) \quad \Gamma_a y = X_a y(a), \quad y \in AC(\mathcal{I}; \mathbb{H}).$$

In accordance with the decomposition (3.9)  $\Gamma_a$  admits the block representation

$$(3.19) \quad \Gamma_a = \left( \Gamma_{0a} : \widehat{\Gamma}_a : \Gamma_{1a} \right)^\top : AC(\mathcal{I}; \mathbb{H}) \rightarrow H \oplus \widehat{H} \oplus H.$$

The particular case of the operator  $X_a$  is (cf. [20])

$$(3.20) \quad X_a = \begin{pmatrix} X_{00} & 0 & X_{01} \\ 0 & I & 0 \\ X_{10} & 0 & X_{11} \end{pmatrix} : H \oplus \widehat{H} \oplus H \rightarrow H \oplus \widehat{H} \oplus H,$$

where the entries  $X_{jk}$  satisfy

$$\text{Im}(X_{00} X_{01}^*) = 0, \quad \text{Im}(X_{10} X_{11}^*) = 0, \quad -X_{10} X_{01}^* + X_{11} X_{00}^* = I_H.$$

If  $X_a$  is given by (3.20) and the function  $y \in AC(\mathcal{I}; \mathbb{H})$  is decomposed as

$$y(t) = \{y_0(t), \widehat{y}(t), y_1(t)\} \in H \oplus \widehat{H} \oplus H, \quad t \in \mathcal{I},$$

then in the representation (3.19) one has

$$(3.21) \quad \Gamma_{0a} y = X_{00} y_0(a) + X_{01} y_1(a), \quad \widehat{\Gamma}_a y = \widehat{y}(a), \quad \Gamma_{1a} y = X_{10} y_0(a) + X_{11} y_1(a).$$



Let  $\lambda \in \mathbb{C}$ . By using the operator  $X_a$  we associate with each operator solution  $Y(\cdot, \lambda) : \mathcal{I} \rightarrow [\mathcal{K}, \mathbb{H}]$  of the equation (3.3) the operator  $Y_a(\lambda) \in [\mathcal{K}, \mathbb{H}]$  given by

$$(3.22) \quad Y_a(\lambda) = X_a Y(a, \lambda)$$

(recall that here  $\mathcal{K}$  is a finite-dimensional Hilbert space).

**Lemma 3.2.** 1) If  $Y(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathcal{K}, \mathbb{H}]$  is an operator solution of Eq. (3.3), then the relation

$$(3.23) \quad \mathcal{K} \ni h \rightarrow (Y(\lambda)h)(t) = Y(t, \lambda)h \in \mathcal{N}_\lambda.$$

defines the linear map  $Y(\lambda) : \mathcal{K} \rightarrow \mathcal{N}_\lambda$  and, conversely, for each such a map  $Y(\lambda)$  there exists a unique operator solution  $Y(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathcal{K}, \mathbb{H}]$  of Eq. (3.3) such that (3.23) holds.

2) Let  $Y(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathcal{K}, \mathbb{H}]$  be an operator solution of Eq. (3.3) and let  $F(\lambda) = \pi Y(\lambda) \in [\mathcal{K}, L_\Delta^2(\mathcal{I})]$ . Then for each  $\tilde{f} \in L_\Delta^2(\mathcal{I})$

$$(3.24) \quad F^*(\lambda)\tilde{f} = \int_{\mathcal{I}} Y^*(t, \lambda)\Delta(t)f(t) dt, \quad f \in \tilde{f}.$$

The first statement of this lemma is obvious, while the second one can be proved in the same way as formula (3.70) in [35].

Clearly, for each solution  $Y(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathcal{K}, \mathbb{H}]$  of Eq. (3.3) the operator (3.22) admits the representation

$$(3.25) \quad Y_a(\lambda) = \Gamma_a Y(\lambda),$$

where  $Y(\lambda)$  is defined in Lemma 3.2.

*Remark 3.3.* According to [38, Remark 5.2] one can construct the map  $\Gamma_b$  by using the following assertion:

— there exist systems of functions  $\{\psi_j\}_1^{\nu_b}$ ,  $\{\varphi_j\}_1^{\hat{\nu}_b}$  and  $\{\theta_j\}_1^{\nu_b}$  in  $\text{dom } \mathcal{T}_{max}$  with  $\nu_b = \min\{\nu_{b+}, \nu_{b-}\}$  and  $\hat{\nu}_b = |\nu_{b+} - \nu_{b-}|$  such that the operators

$$(3.26) \quad \Gamma_{0b}y = \{[y, \psi_j]_b\}_1^{\nu_b}, \quad \hat{\Gamma}_b y = \{[y, \varphi_j]_b\}_1^{\hat{\nu}_b}, \quad \Gamma_{1b}y = \{[y, \theta_j]_b\}_1^{\nu_b}, \quad y \in \text{dom } \mathcal{T}_{max}$$

form the surjective linear map  $\Gamma_b = (\Gamma_{0b} : \hat{\Gamma}_b : \Gamma_{1b})^\top : \text{dom } \mathcal{T}_{max} \rightarrow \mathbb{C}^{\nu_b} \oplus \mathbb{C}^{\hat{\nu}_b} \oplus \mathbb{C}^{\nu_b}$  satisfying the equality (3.16).

This assertion shows that  $\Gamma_b y$  is, in fact, a singular boundary value of a function  $y \in \text{dom } \mathcal{T}_{max}$  (c.f. [9, Ch. 13.2]).

**3.3. Decomposing boundary triplets.** As is known (see for instance [29]) the maximal relation  $T_{max}$  induced by the definite symmetric system (3.2) possesses the following property: for each  $\{\tilde{y}, \tilde{f}\} \in T_{max}$  there exist a unique function  $y \in AC(\mathcal{I}; \mathbb{H}) \cap \mathcal{L}_\Delta^2(\mathcal{I})$  such that  $y \in \tilde{y}$  and  $\{y, f\} \in \mathcal{T}_{max}$  for each  $f \in \tilde{f}$ . Below, without any additional comments, we associate such a function  $y \in AC(\mathcal{I}; \mathbb{H}) \cap \mathcal{L}_\Delta^2(\mathcal{I})$  with each pair  $\{\tilde{y}, \tilde{f}\} \in T_{max}$ .

Let as before  $\Gamma_b$  and  $\Gamma_a$  be the operators (3.15) and (3.19) respectively and let  $H_0$  be the Hilbert (3.11). Consider the following three alternative cases:

Case 1:  $\nu_{b+} - \nu_{b-} \geq \nu_- - \nu_+ \geq 0$ .

It follows from (3.17) that in this case

$$(3.27) \quad \dim \mathcal{H}_b = \nu_{b-}, \quad \dim \hat{\mathcal{H}}_b = \nu_{b+} - \nu_{b-}$$

and (3.10) gives  $\dim \hat{\mathcal{H}}_b \geq \dim \hat{H}$ . Therefore without loss of generality we can assume that  $\hat{H} \subset \hat{\mathcal{H}}_b$  and hence

$$(3.28) \quad \hat{\mathcal{H}}_b = \mathcal{H}'_2 \oplus \hat{H}$$

with  $\mathcal{H}'_2 = \widehat{\mathcal{H}}_b \ominus \widehat{H}$ . Let  $\widetilde{\mathcal{H}}_b = \mathcal{H}_b \oplus \mathcal{H}'_2$  (so that  $\mathcal{H}_b \subset \widetilde{\mathcal{H}}_b$ ) and let

$$(3.29) \quad \widetilde{\Gamma}_{0b} = \Gamma_{0b} + P_{\mathcal{H}'_2} \widehat{\Gamma}_b : \text{dom } \mathcal{T}_{max} \rightarrow \widetilde{\mathcal{H}}_b$$

In *Case 1* we put

$$(3.30) \quad \mathcal{H}_0 = H_0 \oplus \widetilde{\mathcal{H}}_b, \quad \mathcal{H}_1 = H_0 \oplus \mathcal{H}_b,$$

$$(3.31) \quad \begin{aligned} \Gamma'_0 &= \begin{pmatrix} -\Gamma_{1a} + i(\widehat{\Gamma}_a - P_{\widehat{H}} \widehat{\Gamma}_b) \\ \widetilde{\Gamma}_{0b} \end{pmatrix} : \text{dom } \mathcal{T}_{max} \rightarrow H_0 \oplus \widetilde{\mathcal{H}}_b, \\ \Gamma'_1 &= \begin{pmatrix} \Gamma_{0a} + \frac{1}{2}(\widehat{\Gamma}_a + P_{\widehat{H}} \widehat{\Gamma}_b) \\ -\Gamma_{1b} \end{pmatrix} : \text{dom } \mathcal{T}_{max} \rightarrow H_0 \oplus \mathcal{H}_b, \end{aligned}$$

If in addition  $n_+ = n_-$ , then in view of (3.14) and (3.17)  $\widehat{\mathcal{H}}_b = \widehat{H}$  and  $\mathcal{H}'_2 = \{0\}$ . Therefore

$$(3.32) \quad \widetilde{\mathcal{H}}_b = \mathcal{H}_b, \quad \widetilde{\Gamma}_{0b} = \Gamma_{0b}$$

and the equalities (3.30) and (3.31) take the form

$$(3.33) \quad \mathcal{H} = H_0 \oplus \mathcal{H}_b (= \mathcal{H}_0 = \mathcal{H}_1),$$

$$(3.34) \quad \begin{aligned} \Gamma'_0 &= (-\Gamma_{1a} + i(\widehat{\Gamma}_a - \widehat{\Gamma}_b) : \Gamma_{0b})^\top : \text{dom } \mathcal{T}_{max} \rightarrow H_0 \oplus \mathcal{H}_b, \\ \Gamma'_1 &= (\Gamma_{0a} + \frac{1}{2}(\widehat{\Gamma}_a + \widehat{\Gamma}_b) : -\Gamma_{1b})^\top : \text{dom } \mathcal{T}_{max} \rightarrow H_0 \oplus \mathcal{H}_b. \end{aligned}$$

Case 2:  $\nu_- - \nu_+ > \nu_{b+} - \nu_{b-} > 0$ ,

so that the equalities (3.27) holds. It follows from (3.10) that in this case  $\dim \widehat{H} > \dim \widehat{\mathcal{H}}_b$ . Therefore one may assume that  $\widehat{\mathcal{H}}_b \subset \widehat{H}$  and hence  $\widehat{H} = \widehat{\mathcal{H}}_b \oplus \mathcal{H}'_2$  with  $\mathcal{H}'_2 = \widehat{H} \ominus \widehat{\mathcal{H}}_b$ . This implies that the Hilbert space (3.11) admits the representation

$$(3.35) \quad H_0 = H \oplus \underbrace{\widehat{\mathcal{H}}_b \oplus \mathcal{H}'_2}_{\widehat{H}} = H'_0 \oplus \mathcal{H}'_2,$$

where

$$(3.36) \quad H'_0 = H \oplus \widehat{\mathcal{H}}_b.$$

In *Case 2* we let

$$(3.37) \quad \mathcal{H}_0 = H'_0 \oplus \mathcal{H}'_2 \oplus \mathcal{H}_b, \quad \mathcal{H}_1 = H'_0 \oplus \mathcal{H}_b,$$

$$(3.38) \quad \begin{aligned} \Gamma'_0 &= \begin{pmatrix} -\Gamma_{1a} + i(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a - \widehat{\Gamma}_b) \\ iP_{\mathcal{H}'_2} \widehat{\Gamma}_a \\ \Gamma_{0b} \end{pmatrix} : \text{dom } \mathcal{T}_{max} \rightarrow H'_0 \oplus \mathcal{H}'_2 \oplus \mathcal{H}_b, \\ \Gamma'_1 &= \begin{pmatrix} \Gamma_{0a} + \frac{1}{2}(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a + \widehat{\Gamma}_b) \\ -\Gamma_{1b} \end{pmatrix} : \text{dom } \mathcal{T}_{max} \rightarrow H'_0 \oplus \mathcal{H}_b. \end{aligned}$$

Case 3:  $\widehat{\nu} \geq 0 \geq \nu_{b+} - \nu_{b-}$  and  $\widehat{\nu} \neq \nu_{b+} - \nu_{b-} (\neq 0)$ ,

so that in view of (3.17)

$$(3.39) \quad \dim \mathcal{H}_b = \nu_{b+}, \quad \dim \widehat{\mathcal{H}}_b = \nu_{b-} - \nu_{b+}.$$

Let  $\widetilde{\mathcal{H}}_b := \mathcal{H}_b \oplus \widehat{\mathcal{H}}_b$  (so that  $\mathcal{H}_b \subset \widetilde{\mathcal{H}}_b$ ) and let  $\widetilde{\Gamma}_{0b} : \text{dom } \mathcal{T}_{max} \rightarrow \widetilde{\mathcal{H}}_b$  be the linear map given by

$$(3.40) \quad \widetilde{\Gamma}_{0b} = \Gamma_{0b} + \widehat{\Gamma}_b.$$

In *Case 3* we put

$$(3.41) \quad \mathcal{H}_0 = H \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b = H_0 \oplus \widetilde{\mathcal{H}}_b, \quad \mathcal{H}_1 = H \oplus \mathcal{H}_b,$$

$$(3.42) \quad \Gamma'_0 = \begin{pmatrix} -\Gamma_{1a} \\ i\widehat{\Gamma}_a \\ \widetilde{\Gamma}_{0b} \end{pmatrix} : \text{dom } \mathcal{T}_{max} \rightarrow H \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b, \quad \Gamma'_1 = \begin{pmatrix} \Gamma_{0a} \\ -\Gamma_{1b} \end{pmatrix} : \text{dom } \mathcal{T}_{max} \rightarrow H \oplus \mathcal{H}_b.$$

Note that for every system (3.2) one (and only one) of *Cases 1–3* holds. In each of these cases  $\mathcal{H}_1$  is a subspace in  $\mathcal{H}_0$  and  $\Gamma'_j$  is a linear map from  $\text{dom } \mathcal{T}_{max}$  to  $\mathcal{H}_j$ ,  $j \in \{0, 1\}$ . Moreover, the subspace  $\mathcal{H}_2 = \mathcal{H}_0 \ominus \mathcal{H}_1$  coincides with  $\mathcal{H}'_2$  in *Cases 1–2* and  $\mathcal{H}_2 = \widehat{H} \oplus \widetilde{\mathcal{H}}_b$  in *Case 3*. Observe also that according to [38, Proposition 5.5] the deficiency indices of  $T_{min}$  are  $n_{\pm} = \nu_{\pm} + \nu_{b\pm}$ . Therefore  $n_- \leq n_+$  in *Case 1* and  $n_+ < n_-$  in *Cases 2* and *3*. Moreover, formulas (3.10), (3.27) and (3.39) imply that in all *Cases 1–3*

$$(3.43) \quad \dim \mathcal{H}_0 + \dim \mathcal{H}_1 = \nu_+ + \nu_- + \nu_{b+} + \nu_{b-} = n_+ + n_-.$$

**Proposition 3.4.** *Let  $\mathcal{H}_j$  be Hilbert spaces and  $\Gamma'_j : \text{dom } \mathcal{T}_{max} \rightarrow \mathcal{H}_j$ ,  $j \in \{0, 1\}$ , be linear mappings constructed for the alternative *Cases 1–3* before the proposition and let  $\Gamma_j : T_{max} \rightarrow \mathcal{H}_j$ ,  $j \in \{0, 1\}$ , be the operators given by*

$$(3.44) \quad \Gamma_0\{\widetilde{y}, \widetilde{f}\} = \Gamma'_0 y, \quad \Gamma_1\{\widetilde{y}, \widetilde{f}\} = \Gamma'_1 y, \quad \{\widetilde{y}, \widetilde{f}\} \in T_{max}.$$

*Then the collection  $\Pi_{\alpha} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  with  $\alpha = +1$  in *Case 1* and  $\alpha = -1$  in *Cases 2* and *3* is a boundary triplet for  $T_{max}$ .*

*If in addition  $n_+ = n_-$  (so that *Case 1* holds), then  $\Pi_+$  turns into an ordinary boundary triplet  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  for  $T_{max}$ , where  $\mathcal{H}$  is the Hilbert space (3.33) and  $\Gamma_j : T_{max} \rightarrow \mathcal{H}$ ,  $j \in \{0, 1\}$ , are the operators given by (3.44) and (3.34).*

*Proof.* The immediate calculations with taking (3.16) into account show that in each of the *Cases 1–3* the operators  $\Gamma'_0$  and  $\Gamma'_1$  satisfy the relation

$$[y, z]_b - (Jy(a), z(a)) = (\Gamma'_1 y, \Gamma'_0 z) - (\Gamma'_0 y, \Gamma'_1 z) + i\alpha(P_2 \Gamma'_0 y, P_2 \Gamma'_0 z), \quad y, z \in \text{dom } \mathcal{T}_{max}.$$

This and the Lagrange's identity (3.5) give the identity (2.21) for the operators  $\Gamma_0$  and  $\Gamma_1$  defined by (3.44). To prove surjectivity of the mapping  $\Gamma = (\Gamma_0 : \Gamma_1)^{\top}$  note that  $\ker \Gamma'_0 \cap \ker \Gamma'_1 = \ker \Gamma_a \cap \ker \Gamma_b = \text{dom } \mathcal{T}_{min}$ . Hence  $\ker \Gamma (= \ker \Gamma_0 \cap \ker \Gamma_1) = T_{min}$  and by using (3.43) one obtains

$$\dim(\text{dom } \Gamma / \ker \Gamma) = \dim(T_{max} / T_{min}) = n_+ + n_- = \dim(\mathcal{H}_0 \oplus \mathcal{H}_1).$$

This implies that  $\text{ran } \Gamma = \mathcal{H}_0 \oplus \mathcal{H}_1$  and, consequently,  $\Pi_{\alpha}$  is a boundary triplet for  $T_{max}$ .

The latter statement of the proposition follows from reasonings before formula (3.33).  $\square$

**Definition 3.5.** The boundary triplet  $\Pi_{\alpha} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  constructed in Proposition 3.4 will be called a decomposing boundary triplet for  $T_{max}$ .

*Remark 3.6.* In the paper [38] decomposing boundary triplets  $\Pi_+$  were constructed for the maximal relations  $T_{max}$  satisfying the condition  $n_- \leq n_+$ . In *Case 1* such a triplet coincides with the triplet  $\Pi_+$  introduced in Proposition 3.4.

Combining Propositions 3.4 and 2.12 we arrive at the following three propositions.

**Proposition 3.7.** *Let in *Case 1*  $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.31), (3.44) for  $T_{max}$ . Then:*

1) The equalities

$$(3.45) \quad T = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = P_{\hat{H}}\hat{\Gamma}_by, \tilde{\Gamma}_{0b}y = \Gamma_{1b}y = 0\}$$

$$(3.46) \quad T^* = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = P_{\hat{H}}\hat{\Gamma}_by\}$$

define a symmetric extension  $T$  of  $T_{min}$  and its adjoint  $T^*$ . Moreover, the deficiency indices of  $T$  are  $n_+(T) = \nu_{b+} - \hat{\nu}$  and  $n_-(T) = \nu_{b-}$ .

2) The collection  $\dot{\Pi}_+ = \{\tilde{\mathcal{H}}_b \oplus \mathcal{H}_b, \dot{\Gamma}_0, \dot{\Gamma}_1\}$  with the operators

$$(3.47) \quad \dot{\Gamma}_0\{\tilde{y}, \tilde{f}\} = \tilde{\Gamma}_{0b}y, \quad \dot{\Gamma}_1\{\tilde{y}, \tilde{f}\} = -\Gamma_{1b}y, \quad \{\tilde{y}, \tilde{f}\} \in T^*,$$

is a boundary triplet for  $T^*$  and the (maximal symmetric) relation  $A_0 (= \ker \dot{\Gamma}_0)$  is of the form

$$(3.48) \quad A_0 = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = P_{\hat{H}}\hat{\Gamma}_by, \tilde{\Gamma}_{0b}y = 0\}.$$

If in addition  $n_+ = n_-$  and  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  is an ordinary decomposing boundary triplet (3.34), (3.44) for  $T_{max}$ , then the equality (3.45) take the form

$$(3.49) \quad T = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = \hat{\Gamma}_by, \Gamma_{0b}y = \Gamma_{1b}y = 0\}$$

and  $n_+(T) = n_-(T) = \nu_{b-}$ . Moreover, in this case  $A_0 = A_0^*$  and

$$(3.50) \quad A_0 = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = \hat{\Gamma}_by, \Gamma_{0b}y = 0\}.$$

**Proposition 3.8.** Let Case 2 holds and let  $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.38), (3.44) for  $T_{max}$ . Then:

1) Statement 1) of Proposition 3.7 holds with

$$(3.51) \quad T = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = \hat{\Gamma}_by, \Gamma_{0b}y = \Gamma_{1b}y = 0\}$$

$$(3.52) \quad T^* = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, P_{\hat{\mathcal{H}}_b}\hat{\Gamma}_ay = \hat{\Gamma}_by\}.$$

Moreover, the deficiency indices of  $T$  are  $n_+(T) = \nu_{b-}$  and  $n_-(T) = \hat{\nu} + 2\nu_{b-} - \nu_{b+}$ .

2) The collection  $\dot{\Pi}_- = \{(\mathcal{H}'_2 \oplus \mathcal{H}_b) \oplus \mathcal{H}_b, \dot{\Gamma}_0, \dot{\Gamma}_1\}$  with the operators

$$(3.53) \quad \dot{\Gamma}_0\{\tilde{y}, \tilde{f}\} = \{iP_{\mathcal{H}'_2}\hat{\Gamma}_ay, \Gamma_{0b}y\}(\in \mathcal{H}'_2 \oplus \mathcal{H}_b), \quad \dot{\Gamma}_1\{\tilde{y}, \tilde{f}\} = -\Gamma_{1b}y(\in \mathcal{H}_b), \quad \{\tilde{y}, \tilde{f}\} \in T^*,$$

is a boundary triplet for  $T^*$  and  $A_0 (= \ker \dot{\Gamma}_0)$  is of the form

$$(3.54) \quad A_0 = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = \hat{\Gamma}_by, \Gamma_{0b}y = 0\}.$$

**Proposition 3.9.** Let in Case 3  $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.42), (3.44) for  $T_{max}$ . Then:

1) Statement 1) of Proposition 3.7 holds with

$$(3.55) \quad T = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = 0, \tilde{\Gamma}_{0b}y = \Gamma_{1b}y = 0\}$$

$$(3.56) \quad T^* = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0\}.$$

Moreover, the deficiency indices of  $T$  are  $n_+(T) = \nu_{b+}$  and  $n_-(T) = \hat{\nu} + \nu_{b-}$ .

2) The collection  $\dot{\Pi}_- = \{(\hat{H} \oplus \tilde{\mathcal{H}}_b) \oplus \mathcal{H}_b, \dot{\Gamma}_0, \dot{\Gamma}_1\}$  with the operators

$$(3.57) \quad \dot{\Gamma}_0\{\tilde{y}, \tilde{f}\} = \{\hat{\Gamma}_ay, \tilde{\Gamma}_{0b}y\}(\in \hat{H} \oplus \tilde{\mathcal{H}}_b), \quad \dot{\Gamma}_1\{\tilde{y}, \tilde{f}\} = -\Gamma_{1b}y(\in \mathcal{H}_b), \quad \{\tilde{y}, \tilde{f}\} \in T^*,$$

is a boundary triplet for  $T^*$  and  $A_0 (= \ker \dot{\Gamma}_0)$  is of the form

$$(3.58) \quad A_0 = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \hat{\Gamma}_ay = 0, \tilde{\Gamma}_{0b}y = 0\}.$$

4.  $\mathcal{L}_\Delta^2$ -SOLUTIONS OF BOUNDARY VALUE PROBLEMS

4.1. **Case 1.** Assume that in *Case 1*  $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  is a decomposing boundary triplet (3.31), (3.44) for  $T_{max}$  and  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_{+1}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  is a collection of holomorphic pairs (2.3). For a given  $f \in \mathcal{L}_\Delta^2(\mathcal{I})$  consider the following boundary value problem:

$$(4.1) \quad Jy' - B(t)y = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},$$

$$(4.2) \quad \Gamma_{1a}y = 0, \quad \hat{\Gamma}_ay = P_{\tilde{H}}\hat{\Gamma}_by, \quad C_0(\lambda)\tilde{\Gamma}_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_+,$$

$$(4.3) \quad \Gamma_{1a}y = 0, \quad \hat{\Gamma}_ay = P_{\tilde{H}}\hat{\Gamma}_by, \quad D_0(\lambda)\tilde{\Gamma}_{0b}y + D_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_-.$$

A function  $y(\cdot, \cdot) : \mathcal{I} \times (\mathbb{C} \setminus \mathbb{R}) \rightarrow \mathbb{H}$  is called a solution of this problem if for each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  the function  $y(\cdot, \lambda)$  belongs to  $AC(\mathcal{I}; \mathbb{H}) \cap \mathcal{L}_\Delta^2(\mathcal{I})$  and satisfies the equation (4.1) a.e. on  $\mathcal{I}$  (so that  $y \in \text{dom } T_{max}$ ) and the boundary conditions (4.2), (4.3).

Application of Theorem 2.14 to the boundary triplet (3.47) yields the following theorem.

**Theorem 4.1.** *Let in Case 1  $T$  be a symmetric relation in  $L_\Delta^2(\mathcal{I})$  defined by (3.45). If  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_{+1}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  is a collection (2.3), then for every  $f \in \mathcal{L}_\Delta^2(\mathcal{I})$  the boundary problem (4.1) - (4.3) has a unique solution  $y(t, \lambda) = y_f(t, \lambda)$  and the equality*

$$(4.4) \quad R(\lambda)\tilde{f} = \pi(y_f(\cdot, \lambda)), \quad \tilde{f} \in L_\Delta^2(\mathcal{I}), \quad f \in \tilde{f}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

*defines a generalized resolvent  $R(\lambda) =: R_\tau(\lambda)$  of  $T$ . Conversely, for each generalized resolvent  $R(\lambda)$  of  $T$  there exists a unique  $\tau \in \tilde{R}_{+1}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  such that  $R(\lambda) = R_\tau(\lambda)$ .*

If  $n_+ = n_-$ , then (3.32) is valid. This and Theorem 4.1 yield the following corollary.

**Corollary 4.2.** *Let  $n_+ = n_-$ , let  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.34), (3.44) for  $T_{max}$  and let  $T$  be a symmetric relation (3.49). Then the statements of Theorem 4.1 hold with the Nevanlinna operator pairs  $\tau \in \tilde{R}(\mathcal{H}_b)$  in the form (2.16) and the following boundary conditions in place of (4.2) and (4.3):*

$$(4.5) \quad \Gamma_{1a}y = 0, \quad \hat{\Gamma}_ay = \hat{\Gamma}_by, \quad C_0(\lambda)\Gamma_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

*In this case  $R_\tau(\lambda)$  is a canonical resolvent of  $T$  if and only if  $\tau \in \tilde{R}^0(\mathcal{H}_b)$ .*

**Remark 4.3.** Let in Theorem 4.1  $\tau_0 = \{\tau_+, \tau_-\} \in \tilde{R}_{+1}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  be defined by (2.3) with

$$(4.6) \quad C_0(\lambda) \equiv I_{\tilde{\mathcal{H}}_b}, \quad C_1(\lambda) \equiv 0 \quad \text{and} \quad D_0(\lambda) \equiv P_{\mathcal{H}_b}(\in [\tilde{\mathcal{H}}_b, \mathcal{H}_b]), \quad D_1(\lambda) \equiv 0$$

and let  $R_0(\lambda) = R_{\tau_0}(\lambda)$  be the corresponding generalized resolvent of  $T$ . Then

$$R_0(\lambda) = (A_0 - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+ \quad \text{and} \quad R_0(\lambda) = (A_0^* - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_-,$$

where  $A_0$  is given by (3.48).

Similarly, let in Corollary 4.2  $\tau_0 = \{(I_{\mathcal{H}_b}, 0); \mathcal{H}_b\} \in \tilde{R}^0(\mathcal{H}_b)$ . Then  $R_0(\lambda) := R_{\tau_0}(\lambda) = (A_0 - \lambda)^{-1}$ , where  $A_0$  is the selfadjoint extension (3.50).

**Proposition 4.4.** *Let in Case 1  $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.31), (3.44) for  $T_{max}$ , let  $\gamma_\pm(\cdot)$  be the corresponding  $\gamma$ -fields and let*

$$(4.7) \quad M_+(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix} : H_0 \oplus \tilde{\mathcal{H}}_b \rightarrow H_0 \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_+$$

$$(4.8) \quad M_-(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2-}(\lambda) \\ M_{3-}(\lambda) & M_{4-}(\lambda) \end{pmatrix} : H_0 \oplus \mathcal{H}_b \rightarrow H_0 \oplus \tilde{\mathcal{H}}_b, \quad \lambda \in \mathbb{C}_-$$

*be the block representations of the corresponding Weyl functions. Then:*

1) For every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists an operator solution  $v_0(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  of Eq. (3.3) such that

$$(4.9) \quad \Gamma_{1a}v_0(\lambda) = -P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(4.10) \quad (\Gamma_{0a} + \widehat{\Gamma}_a)v_0(\lambda) = m_0(\lambda) - \frac{i}{2}P_{\widehat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(4.11) \quad i(\widehat{\Gamma}_a - P_{\widehat{H}}\widehat{\Gamma}_b)v_0(\lambda) = P_{\widehat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(4.12) \quad \widetilde{\Gamma}_{0b}v_0(\lambda) = 0, \quad \Gamma_{1b}v_0(\lambda) = -M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+,$$

$$(4.13) \quad \widetilde{\Gamma}_{0b}v_0(\lambda) = -iP_{\mathcal{H}'_2}M_{3-}(\lambda), \quad \Gamma_{1b}v_0(\lambda) = -P_{\mathcal{H}_b}M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-.$$

2) For every  $\lambda \in \mathbb{C}_+$  ( $\lambda \in \mathbb{C}_-$ ) there exists a solution  $u_+(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\widetilde{\mathcal{H}}_b, \mathbb{H}]$  (resp.  $u_-(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathcal{H}_b, \mathbb{H}]$ ) such that

$$(4.14) \quad \Gamma_{1a}u_\pm(\lambda) = 0, \quad \lambda \in \mathbb{C}_\pm,$$

$$(4.15) \quad (\Gamma_{0a} + \widehat{\Gamma}_a)u_\pm(\lambda) = M_{2\pm}(\lambda), \quad \lambda \in \mathbb{C}_\pm,$$

$$(4.16) \quad i(\widehat{\Gamma}_a - P_{\widehat{H}}\widehat{\Gamma}_b)u_\pm(\lambda) = 0, \quad \lambda \in \mathbb{C}_\pm,$$

$$(4.17) \quad \widetilde{\Gamma}_{0b}u_+(\lambda) = I_{\widetilde{\mathcal{H}}_b}, \quad \Gamma_{1b}u_+(\lambda) = -M_{4+}(\lambda), \quad \lambda \in \mathbb{C}_+,$$

$$(4.18) \quad \widetilde{\Gamma}_{0b}u_-(\lambda) = I_{\mathcal{H}_b} - iP_{\mathcal{H}'_2}M_{4-}(\lambda), \quad \Gamma_{1b}u_-(\lambda) = -P_{\mathcal{H}_b}M_{4-}(\lambda), \quad \lambda \in \mathbb{C}_-.$$

In formulas (4.9)–(4.18)  $v_0(\lambda)$  and  $u_\pm(\lambda)$  are linear maps from Lemma 3.2 corresponding to the solutions  $v_0(\cdot, \lambda)$  and  $u_\pm(\cdot, \lambda)$  respectively.

3) The solutions  $v_0(\cdot, \lambda)$  and  $u_\pm(\cdot, \lambda)$  are connected with  $\gamma$ -fields  $\gamma_\pm(\cdot)$  by

$$(4.19) \quad \gamma_\pm(\lambda) \upharpoonright H_0 = \pi v_0(\lambda), \quad \lambda \in \mathbb{C}_\pm;$$

$$(4.20) \quad \gamma_+(\lambda) \upharpoonright \widetilde{\mathcal{H}}_b = \pi u_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(\lambda) \upharpoonright \mathcal{H}_b = \pi u_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

*Proof.* Let  $\gamma_\pm(\cdot)$  be the  $\gamma$ -fields (2.25) of the triplet  $\Pi_+$ . Since the quotient mapping  $\pi$  isomorphically maps  $\mathcal{N}_\lambda$  onto  $\mathfrak{N}_\lambda(T_{min})$ , it follows that for every  $\lambda \in \mathbb{C}_+$  ( $\lambda \in \mathbb{C}_-$ ) there exists an isomorphism  $Z_+(\lambda) : \mathcal{H}_0 \rightarrow \mathcal{N}_\lambda$  (resp.  $Z_-(\lambda) : \mathcal{H}_1 \rightarrow \mathcal{N}_\lambda$ ) such that

$$(4.21) \quad \gamma_+(\lambda) = \pi Z_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(\lambda) = \pi Z_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

Combining of (4.21) with (2.25) – (2.27) and the obvious equality  $\Gamma_j\{\pi y, \lambda \pi y\} = \Gamma'_j y$ ,  $y \in \mathcal{N}_\lambda$ ,  $j \in \{0, 1\}$ , gives

$$(4.22) \quad \Gamma'_0 Z_+(\lambda) = I_{\mathcal{H}_0}, \quad \Gamma'_1 Z_+(\lambda) = M_+(\lambda), \quad \lambda \in \mathbb{C}_+,$$

$$(4.23) \quad P_{\mathcal{H}_1} \Gamma'_0 Z_-(\lambda) = I_{\mathcal{H}_1}, \quad (\Gamma'_1 + iP_{\mathcal{H}'_2} \Gamma'_0) Z_-(\lambda) = M_-(\lambda), \quad \lambda \in \mathbb{C}_-,$$

which in view of (3.31) can be written as

$$(4.24) \quad \begin{pmatrix} -\Gamma_{1a} + i(\widehat{\Gamma}_a - P_{\widehat{H}}\widehat{\Gamma}_b) \\ \widetilde{\Gamma}_{0b} \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} I_{H_0} & 0 \\ 0 & I_{\widetilde{\mathcal{H}}_b} \end{pmatrix}, \quad \lambda \in \mathbb{C}_+$$

$$(4.25) \quad \begin{pmatrix} \Gamma_{0a} + \frac{1}{2}(\widehat{\Gamma}_a + P_{\widehat{H}}\widehat{\Gamma}_b) \\ -\Gamma_{1b} \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_+$$

$$(4.26) \quad \begin{pmatrix} -\Gamma_{1a} + i(\widehat{\Gamma}_a - P_{\widehat{H}}\widehat{\Gamma}_b) \\ \Gamma_{0b} \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} I_{H_0} & 0 \\ 0 & I_{\mathcal{H}_b} \end{pmatrix}, \quad \lambda \in \mathbb{C}_-$$

$$(4.27) \quad \begin{pmatrix} \Gamma_{0a} + \frac{1}{2}(\widehat{\Gamma}_a + P_{\widehat{H}}\widehat{\Gamma}_b) \\ -\Gamma_{1b} + iP_{\mathcal{H}'_2}\Gamma_b \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2-}(\lambda) \\ M_{3-}(\lambda) & M_{4-}(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_-.$$

It follows from (4.24)–(4.27) that

$$(4.28) \quad \Gamma_{1a}Z_{\pm}(\lambda) = (-P_H : 0), \quad \frac{1}{2}(\widehat{\Gamma}_a - P_{\widehat{H}}\widehat{\Gamma}_b)Z_{\pm}(\lambda) = (-\frac{i}{2}P_{\widehat{H}} : 0), \quad \lambda \in \mathbb{C}_{\pm}$$

$$(4.29) \quad \Gamma_{0a}Z_{\pm}(\lambda) = (P_H m_0(\lambda) : P_H M_{2\pm}(\lambda)), \quad \lambda \in \mathbb{C}_{\pm}$$

$$(4.30) \quad \frac{1}{2}(\widehat{\Gamma}_a + P_{\widehat{H}}\widehat{\Gamma}_b)Z_{\pm}(\lambda) = (P_{\widehat{H}} m_0(\lambda) : P_{\widehat{H}} M_{2\pm}(\lambda)), \quad \lambda \in \mathbb{C}_{\pm}.$$

Summing up the second equality in (4.28) with (4.29) and (4.30) one obtains

$$(4.31) \quad (\Gamma_{0a} + \widehat{\Gamma}_a)Z_{\pm}(\lambda) = (m_0(\lambda) - \frac{i}{2}P_{\widehat{H}} : M_{2\pm}(\lambda)), \quad \lambda \in \mathbb{C}_{\pm}.$$

Moreover, (4.24)–(4.27) yield

$$(4.32) \quad \widetilde{\Gamma}_{0b}Z_+(\lambda) = (0 : I_{\widetilde{\mathcal{H}}_b}), \quad \Gamma_{1b}Z_+(\lambda) = (-M_{3+}(\lambda) : -M_{4+}(\lambda)), \quad \lambda \in \mathbb{C}_+,$$

$$(4.33) \quad \Gamma_{1b}Z_-(\lambda) = (-P_{\mathcal{H}_b} M_{3-}(\lambda) : -P_{\mathcal{H}_b} M_{4-}(\lambda)), \quad \lambda \in \mathbb{C}_-,$$

$$\Gamma_{0b}Z_-(\lambda) = (0 : I_{\mathcal{H}_b}), \quad P_{\mathcal{H}'_2}\widehat{\Gamma}_b Z_-(\lambda) = (-iP_{\mathcal{H}'_2} M_{3-}(\lambda) : -iP_{\mathcal{H}'_2} M_{4-}(\lambda)), \quad \lambda \in \mathbb{C}_-$$

and in view of (3.29) one has

$$(4.34) \quad \widetilde{\Gamma}_{0b}Z_-(\lambda) = (-iP_{\mathcal{H}'_2} M_{3-}(\lambda) : I_{\mathcal{H}_b} - iP_{\mathcal{H}'_2} M_{4-}(\lambda)), \quad \lambda \in \mathbb{C}_-.$$

Assume now that the block representations of  $Z_{\pm}(\lambda)$  are

$$(4.35) \quad Z_+(\lambda) = (v_0(\lambda) : u_+(\lambda)) : H_0 \oplus \widetilde{\mathcal{H}}_b \rightarrow \mathcal{N}_{\lambda}, \quad \lambda \in \mathbb{C}_+$$

$$(4.36) \quad Z_-(\lambda) = (v_0(\lambda) : u_-(\lambda)) : H_0 \oplus \mathcal{H}_b \rightarrow \mathcal{N}_{\lambda}, \quad \lambda \in \mathbb{C}_-$$

and let  $v_0(\cdot, \lambda) \in \mathcal{L}_{\Delta}^2[H_0, \mathbb{H}]$ ,  $u_+(\cdot, \lambda) \in \mathcal{L}_{\Delta}^2[\widetilde{\mathcal{H}}_b, \mathbb{H}]$  and  $u_-(\cdot, \lambda) \in \mathcal{L}_{\Delta}^2[\mathcal{H}_b, \mathbb{H}]$  be the operator solutions of Eq. (3.3) corresponding to  $v_0(\lambda)$ ,  $u_+(\lambda)$  and  $u_-(\lambda)$  respectively (see Lemma 3.2). Then the representations (4.35) and (4.36) together with (4.28), (4.31) and (4.32) – (4.34) yield the relations (4.9)–(4.18) for  $v_0(\cdot, \lambda)$  and  $u_{\pm}(\cdot, \lambda)$ .

Finally, (4.19) and (4.20) follow from (4.21) and (4.35), (4.36).  $\square$

**Theorem 4.5.** *Let the assumptions of Proposition 4.4 be satisfied and let  $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}_{+1}(\widetilde{\mathcal{H}}_b, \mathcal{H}_b)$  be a collection of operator pairs (2.3). Then:*

1) *For each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists a unique operator solution  $v_{\tau}(\cdot, \lambda) \in \mathcal{L}_{\Delta}^2[H_0, \mathbb{H}]$  of Eq. (3.3) satisfying the boundary conditions*

$$(4.37) \quad \Gamma_{1a}v_{\tau}(\lambda) = -P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(4.38) \quad i(\widehat{\Gamma}_a - P_{\widehat{H}}\widehat{\Gamma}_b)v_{\tau}(\lambda) = P_{\widehat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(4.39) \quad C_0(\lambda)\widetilde{\Gamma}_{0b}v_{\tau}(\lambda) + C_1(\lambda)\Gamma_{1b}v_{\tau}(\lambda) = 0, \quad \lambda \in \mathbb{C}_+,$$

$$(4.40) \quad D_0(\lambda)\widetilde{\Gamma}_{0b}v_{\tau}(\lambda) + D_1(\lambda)\Gamma_{1b}v_{\tau}(\lambda) = 0, \quad \lambda \in \mathbb{C}_-$$

(here  $P_H$  and  $P_{\widehat{H}}$  are the orthoprojectors in  $H_0$  onto  $H$  and  $\widehat{H}$  respectively).

2)  $v_{\tau}(\cdot, \lambda)$  is connected with the solutions  $v_0(\cdot, \lambda)$  and  $u_{\pm}(\cdot, \lambda)$  from Proposition 4.4 by

$$(4.41) \quad v_{\tau}(t, \lambda) = v_0(t, \lambda) - u_+(t, \lambda)(\tau_+(\lambda) + M_{4+}(\lambda))^{-1}M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+$$

$$(4.42) \quad v_{\tau}(t, \lambda) = v_0(t, \lambda) - u_-(t, \lambda)(\tau_+^*(\overline{\lambda}) + M_{4-}(\lambda))^{-1}M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-.$$

If in addition  $n_+ = n_-$  and  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  is a decomposing boundary triplet (3.34), (3.44) for  $T_{max}$ , then  $\tau \in \widetilde{R}(\mathcal{H}_b)$  is given by (2.16) and the boundary conditions (4.37)–(4.40) take the form

$$\Gamma_{1a}v_{\tau}(\lambda) = -P_H, \quad i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)v_{\tau}(\lambda) = P_{\widehat{H}}, \quad C_0(\lambda)\Gamma_{0b}v_{\tau}(\lambda) + C_1(\lambda)\Gamma_{1b}v_{\tau}(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

*Proof.* Since in view of Proposition 3.7, 2)  $M_{4\pm}(\cdot)$  are the Weyl functions of the boundary triplet  $\dot{\Pi}_+$ , it follows from [33] that  $0 \in \rho(\tau_+(\lambda) + M_{4+}(\lambda))$ ,  $\lambda \in \mathbb{C}_+$ , and  $0 \in \rho(\tau_+^*(\bar{\lambda}) + M_{4-}(\lambda))$ ,  $\lambda \in \mathbb{C}_-$ . Therefore for each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  the equalities (4.41) and (4.42) correctly define the solution  $v_\tau(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  of Eq. (3.3). Let us show that this solution satisfies (4.37)–(4.40).

Combining (4.41) and (4.42) with (4.9), (4.11) and (4.14), (4.16) one gets the equalities (4.37) and (4.38). To prove (4.39) and (4.40) we let  $T_+(\lambda) = (\tau_+(\lambda) + M_{4+}(\lambda))^{-1}$ ,  $\lambda \in \mathbb{C}_+$ , and  $T_-(\lambda) = (\tau_+^*(\bar{\lambda}) + M_{4-}(\lambda))^{-1}$ ,  $\lambda \in \mathbb{C}_-$ . Then

$$(4.43) \quad \tau_+(\lambda) = \{ \{T_+(\lambda)h, (I - M_{4+}(\lambda)T_+(\lambda))h\} : h \in \mathcal{H}_b \}$$

and  $\tau_+^*(\bar{\lambda}) = \{ \{T_-(\lambda)h, h - M_{4-}(\lambda)T_-(\lambda)h\} : h \in \tilde{\mathcal{H}}_b \}$ , which in view of (2.11) yields

$$(4.44) \quad \tau_-(\lambda) = \{ \{(-T_-(\lambda) - iP_{\mathcal{H}'_2} + iP_{\mathcal{H}'_2}M_{4-}(\lambda)T_-(\lambda))h, \\ (-P_{\mathcal{H}_b} + P_{\mathcal{H}_b}M_{4-}(\lambda)T_-(\lambda))h\} : h \in \tilde{\mathcal{H}}_b \}.$$

Moreover, the relations (4.41) and (4.42) with taking (4.12), (4.13), (4.17) and (4.18) into account give

$$\begin{aligned} \tilde{\Gamma}_{0b}v_\tau\lambda &= -T_+(\lambda)M_{3+}(\lambda), \quad \Gamma_{1b}v_\tau\lambda = -(I - M_{4+}(\lambda)T_+(\lambda))M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+, \\ \tilde{\Gamma}_{0b}v_\tau\lambda &= (-iP_{\mathcal{H}'_2} - T_-(\lambda) + iP_{\mathcal{H}'_2}M_{4-}(\lambda)T_-(\lambda))M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-, \\ \Gamma_{1b}v_\tau\lambda &= (-P_{\mathcal{H}_b} + P_{\mathcal{H}_b}M_{4-}(\lambda)T_-(\lambda))M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_- \end{aligned}$$

Hence by (4.43) and (4.44) one has

$$(4.45) \quad \{\tilde{\Gamma}_{0b}v_\tau(\lambda)h_0, \Gamma_{1b}v_\tau(\lambda)h_0\} \in \tau_\pm(\lambda), \quad h_0 \in H_0, \quad \lambda \in \mathbb{C}_\pm,$$

which in view of the equalities (2.3) yields (4.39) and (4.40).

Next assume that  $v_1(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  and  $v_2(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  are the operator solutions of Eq. (3.3) satisfying (4.37)–(4.40) and let  $v(t, \lambda) = v_1(t, \lambda) - v_2(t, \lambda)$ . Then for each  $h_0 \in H_0$  the function  $y = v(t, \lambda)h_0$  is a solution of the homogenous boundary problem (4.1)–(4.3) (with  $f = 0$ ). Since by Theorem 4.1 such a problem has a unique solution  $y = 0$ , it follows that  $v(t, \lambda) = 0$ . This proves the uniqueness of  $v_\tau(\cdot, \lambda)$ .  $\square$

**4.2. Case 2.** Applying Corollary 2.15 to the boundary triplet (3.53) we obtain the following theorem.

**Theorem 4.6.** *Let in Case 2  $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.38), (3.44) for  $T_{max}$ , let  $T$  be a symmetric relation in  $L_\Delta^2(\mathcal{I})$  defined by (3.51) and let  $\tau \in \tilde{R}(\mathcal{H}_b)$  be a Nevanlinna operator pair (2.16). Then for every  $f \in \mathcal{L}_\Delta^2(\mathcal{I})$  the boundary value problem*

$$(4.46) \quad Jy' - B(t)y = \lambda\Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},$$

$$(4.47) \quad \Gamma_{1a}y = 0, \quad P_{\hat{\mathcal{H}}_b}\hat{\Gamma}_ay = \hat{\Gamma}_by, \quad C_0(\lambda)\Gamma_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_+,$$

$$(4.48) \quad \Gamma_{1a}y = 0, \quad \hat{\Gamma}_ay = \hat{\Gamma}_by, \quad C_0(\lambda)\Gamma_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_-.$$

*has a unique solution  $y(t, \lambda) = y_f(t, \lambda)$  (in the same sense as the problem (4.1)–(4.3)) and the equality (4.4) gives a generalized resolvent  $R(\lambda) =: R_\tau(\lambda)$  of  $T$ .*

*Remark 4.7.* Let in Theorem 4.6  $\tau_0 = \{(I_{\mathcal{H}_b}, 0); \mathcal{H}_b\} \in \tilde{R}^0(\mathcal{H}_b)$  and let  $A_0$  be the symmetric extension (3.54). Then  $R_0(\lambda) := R_{\tau_0}(\lambda)$  is if the form

$$(4.49) \quad R_0(\lambda) = (A_0^* - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+ \quad \text{and} \quad R_0(\lambda) = (A_0 - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_-,$$



**Proposition 4.8.** *Assume that in Case 2  $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  is a decomposing boundary triplet (3.38), (3.44) for  $T_{max}$ ,  $\gamma_\pm(\cdot)$  are the  $\gamma$ -fields of  $\Pi_-$  and*

$$(4.50) \quad M_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ N_{1+}(\lambda) & N_{2+}(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : H'_0 \oplus \mathcal{H}_b \rightarrow H'_0 \oplus \mathcal{H}'_2 \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_+$$

$$(4.51) \quad M_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) & M_2(\lambda) \\ M_3(\lambda) & N_{2-}(\lambda) & M_4(\lambda) \end{pmatrix} : H'_0 \oplus \mathcal{H}'_2 \oplus \mathcal{H}_b \rightarrow H'_0 \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_-$$

are the block representations of the corresponding Weyl functions. Moreover, let  $m_0(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [H_0]$  be the operator function given by

$$(4.52) \quad m_0(\lambda) = \begin{pmatrix} M_1(\lambda) & 0 \\ N_{1+}(\lambda) & \frac{i}{2}I_{\mathcal{H}'_2} \end{pmatrix} : \underbrace{H'_0 \oplus \mathcal{H}'_2}_{H_0} \rightarrow \underbrace{H'_0 \oplus \mathcal{H}'_2}_{H_0}, \quad \lambda \in \mathbb{C}_+$$

$$(4.53) \quad m_0(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) \\ 0 & -\frac{i}{2}I_{\mathcal{H}'_2} \end{pmatrix} : \underbrace{H'_0 \oplus \mathcal{H}'_2}_{H_0} \rightarrow \underbrace{H'_0 \oplus \mathcal{H}'_2}_{H_0}, \quad \lambda \in \mathbb{C}_-.$$

Then: 1) for every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists an operator solution  $v_0(\cdot, \lambda) \in \mathcal{L}^2_\Delta[H_0, \mathbb{H}]$  of Eq. (3.3) such that (4.9) and (4.10) hold and

$$(4.54) \quad i(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a - \widehat{\Gamma}_b)v_0(\lambda) = P_{\widehat{\mathcal{H}}_b}, \quad \lambda \in \mathbb{C}_+; \quad i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)v_0(\lambda) = P_{\widehat{H}}, \quad \lambda \in \mathbb{C}_-,$$

$$(4.55) \quad \Gamma_{0b}v_0(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(4.56) \quad \Gamma_{1b}v_0(\lambda) = -M_3(\lambda)P_{H'_0}, \quad \lambda \in \mathbb{C}_+; \quad \Gamma_{1b}v_0(\lambda) = (-M_3(\lambda) : -N_{2-}(\lambda)), \quad \lambda \in \mathbb{C}_-$$

(in (4.56)  $P_{H'_0}$  is the orthoprojector in  $H_0$  onto  $H'_0$ ). Moreover, for every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists a solution  $u(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}_b, \mathbb{H}]$  of Eq. (3.3) such that

$$(4.57) \quad \Gamma_{1a}u(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(4.58) \quad (\Gamma_{0a} + \widehat{\Gamma}_a)u(\lambda) = M_2(\lambda) + N_{2+}(\lambda), \quad \lambda \in \mathbb{C}_+; \quad (\Gamma_{0a} + \widehat{\Gamma}_a)u(\lambda) = M_2(\lambda), \quad \lambda \in \mathbb{C}_-$$

$$(4.59) \quad i(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a - \widehat{\Gamma}_b)u(\lambda) = 0, \quad \lambda \in \mathbb{C}_+; \quad i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)u(\lambda) = 0, \quad \lambda \in \mathbb{C}_-,$$

$$(4.60) \quad \Gamma_{0b}u(\lambda) = I_{\mathcal{H}_b}, \quad \Gamma_{1b}u(\lambda) = -M_4(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

2) The following equalities hold

$$(4.61) \quad \gamma_-(\lambda) \upharpoonright H_0 = \pi v_0(\lambda), \quad \lambda \in \mathbb{C}_-; \quad \gamma_\pm(\lambda) \upharpoonright \mathcal{H}_b = \pi u(\lambda), \quad \lambda \in \mathbb{C}_\pm.$$

*Proof.* 1) By using the reasonings from Proposition 4.4 to the  $\gamma$ -fields (2.33) of the triplet  $\Pi_-$  one can prove that for every  $\lambda \in \mathbb{C}_+$  ( $\lambda \in \mathbb{C}_-$ ) there exists an isomorphism  $Z_+(\lambda) : \mathcal{H}_1 \rightarrow \mathcal{N}_\lambda$  (resp.  $Z_-(\lambda) : \mathcal{H}_0 \rightarrow \mathcal{N}_\lambda$ ) such that (4.21) holds and the relations

$$(4.62) \quad P_{\mathcal{H}_1} \Gamma'_0 Z_+(\lambda) = I_{\mathcal{H}_1}, \quad (\Gamma'_1 - iP_{\mathcal{H}'_2} \Gamma'_0) Z_+(\lambda) = M_+(\lambda), \quad \lambda \in \mathbb{C}_+,$$

$$(4.63) \quad \Gamma'_0 Z_-(\lambda) = I_{\mathcal{H}_0}, \quad \Gamma'_1 Z_-(\lambda) = M_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

are valid. In view of (3.38) the equalities (4.62) can be represented as

$$(4.64) \quad \begin{pmatrix} -\Gamma_{1a} + i(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a - \widehat{\Gamma}_b) \\ \Gamma_{0b} \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} I_{H'_0} & 0 \\ 0 & I_{\mathcal{H}_b} \end{pmatrix}, \quad \lambda \in \mathbb{C}_+$$

$$(4.65) \quad \begin{pmatrix} \Gamma_{0a} + \frac{1}{2}(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a + \widehat{\Gamma}_b) \\ P_{\mathcal{H}'_2} \widehat{\Gamma}_a \\ -\Gamma_{1b} \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ N_{1+}(\lambda) & N_{2+}(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_+$$

Therefore for all  $\lambda \in \mathbb{C}_+$  one has

$$(4.66) \quad \Gamma_{1a} Z_+(\lambda) = (-P_H : 0), \quad \frac{1}{2}(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a - \widehat{\Gamma}_b) Z_+(\lambda) = (-\frac{i}{2} P_{\widehat{\mathcal{H}}_b} : 0),$$

$$(4.67) \quad \Gamma_{0a} Z_+(\lambda) = (P_H M_1(\lambda) : P_H M_2(\lambda)),$$

$$(4.68) \quad \frac{1}{2}(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a + \widehat{\Gamma}_b) Z_+(\lambda) = (P_{\widehat{\mathcal{H}}_b} M_1(\lambda) : P_{\widehat{\mathcal{H}}_b} M_2(\lambda)),$$

$$(4.69) \quad P_{\mathcal{H}'_2} \widehat{\Gamma}_a Z_+(\lambda) = (N_{1+}(\lambda) : N_{2+}(\lambda)).$$

Moreover, summing up the second equality in (4.66) with the equalities (4.67), (4.68) and (4.69) one gets

$$(4.70) \quad (\Gamma_{0a} + \widehat{\Gamma}_a) Z_+(\lambda) = (M_1(\lambda) + N_{1+}(\lambda) - \frac{i}{2} P_{\widehat{\mathcal{H}}_b} : M_2(\lambda) + N_{2+}(\lambda)) : H'_0 \oplus \mathcal{H}_b \rightarrow H_0.$$

Next, by using (3.38) we may rewrite the equalities (4.63) as

$$(4.71) \quad \begin{pmatrix} -\Gamma_{1a} + i(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a - \widehat{\Gamma}_b) \\ iP_{\mathcal{H}'_2} \widehat{\Gamma}_a \\ \Gamma_{0b} \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} I_{H'_0} & 0 & 0 \\ 0 & I_{\mathcal{H}'_2} & 0 \\ 0 & 0 & I_{\mathcal{H}_b} \end{pmatrix}, \quad \lambda \in \mathbb{C}_-$$

$$(4.72) \quad \begin{pmatrix} \Gamma_{0a} + \frac{1}{2}(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a + \widehat{\Gamma}_b) \\ -\Gamma_{1b} \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) & M_2(\lambda) \\ M_3(\lambda) & N_{2-}(\lambda) & M_4(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_-.$$

In view of (3.37) and (3.35) one has

$$(4.73) \quad \mathcal{H}_0 = H_0 \oplus \mathcal{H}_b.$$

Let  $T_-(\lambda) \in [H_0, H'_0]$  be the operator given by the block representation

$$(4.74) \quad T_-(\lambda) = (M_1(\lambda) : N_{1-}(\lambda)) : H'_0 \oplus \mathcal{H}'_2 \rightarrow H'_0, \quad \lambda \in \mathbb{C}_-.$$

It follows from (4.71) and (4.72) that in the decomposition (4.73) of  $\mathcal{H}_0$  the following equalities hold for all  $\lambda \in \mathbb{C}_-$ :

$$(4.75) \quad \Gamma_{1a} Z_-(\lambda) = (-P_H : 0), \quad \frac{1}{2}(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a - \widehat{\Gamma}_b) Z_-(\lambda) = (-\frac{i}{2} P_{\widehat{\mathcal{H}}_b} : 0),$$

$$(4.76) \quad P_{\mathcal{H}'_2} \widehat{\Gamma}_a Z_-(\lambda) = (-i P_{\mathcal{H}'_2} : 0), \quad \Gamma_{0a} Z_-(\lambda) = (P_H T_-(\lambda) : P_H M_2(\lambda)),$$

$$(4.77) \quad \frac{1}{2}(P_{\widehat{\mathcal{H}}_b} \widehat{\Gamma}_a + \widehat{\Gamma}_b) Z_-(\lambda) = (P_{\widehat{\mathcal{H}}_b} T_-(\lambda) : P_{\widehat{\mathcal{H}}_b} M_2(\lambda)),$$

Multiplying the second equality in (4.75) by 2 and summing up with the first equality in (4.76) we obtain

$$(4.78) \quad (\widehat{\Gamma}_a - \widehat{\Gamma}_b) Z_-(\lambda) = -i(P_{\widehat{\mathcal{H}}} : 0), \quad \lambda \in \mathbb{C}_-.$$

Moreover, summing up the second equality in (4.75) with the equalities (4.76) and (4.77) one gets

$$(4.79) \quad (\Gamma_{0a} + \widehat{\Gamma}_a) Z_-(\lambda) = (T_-(\lambda) - \frac{i}{2} P_{\widehat{\mathcal{H}}_b} - i P_{\mathcal{H}'_2} : M_2(\lambda)), \quad \lambda \in \mathbb{C}_-.$$

Assume now that

$$(4.80) \quad Z_+(\lambda) = (r(\lambda) : u(\lambda)) : H'_0 \oplus \mathcal{H}_b \rightarrow \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_+$$

$$(4.81) \quad Z_-(\lambda) = (v_0(\lambda) : u(\lambda)) : H_0 \oplus \mathcal{H}_b \rightarrow \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_-$$

are the block representations of  $Z_\pm(\lambda)$  (in (4.81) we make use of the decomposition (4.73) of  $\mathcal{H}_0$ ). Moreover, let

$$(4.82) \quad v_0(\lambda) := (r(\lambda) : 0) : H'_0 \oplus \mathcal{H}'_2 \rightarrow \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_+.$$

Then the equalities (4.81) and (4.82) define the linear mapping  $v_0(\lambda) : H_0 \rightarrow \mathcal{N}_\lambda$ , while (4.80) and (4.81) give the mapping  $u(\lambda) : \mathcal{H}_b \rightarrow \mathcal{N}_\lambda$ . Next we show that the operator solutions  $v_0(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  and  $u(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathcal{H}_b, \mathbb{H}]$  of Eq. (3.3) corresponding to  $v_0(\lambda)$  and  $u(\lambda)$  in accordance with Lemma 3.2 have the desired properties.

Combining (4.80)–(4.82) with (4.66), the first equality in (4.75) and (4.78) we obtain (4.9) and the relations (4.54), (4.57) and (4.59). Moreover, in view of (4.52), (4.53) and (4.74) one has

$$\begin{aligned} m_0(\lambda) - \frac{i}{2}P_{\widehat{H}} &= (M_1(\lambda) + N_{1+}(\lambda))P_{H'_0} + \frac{i}{2}P_{\mathcal{H}'_2} - \frac{i}{2}(P_{\widehat{\mathcal{H}}_b} + P_{\mathcal{H}'_2}) = \\ &= (M_1(\lambda) + N_{1+}(\lambda) - \frac{i}{2}P_{\widehat{\mathcal{H}}_b})P_{H'_0}, \quad \lambda \in \mathbb{C}_+, \\ m_0(\lambda) - \frac{i}{2}P_{\widehat{H}} &= T_-(\lambda) - \frac{i}{2}P_{\mathcal{H}'_2} - \frac{i}{2}(P_{\widehat{\mathcal{H}}_b} + P_{\mathcal{H}'_2}) = T_-(\lambda) - \frac{i}{2}P_{\widehat{\mathcal{H}}_b} - iP_{\mathcal{H}'_2}, \quad \lambda \in \mathbb{C}_-. \end{aligned}$$

Combining these equalities with (4.70), (4.79) and taking the block representations (4.80)–(4.82) into account one gets the equalities (4.10) and (4.58). Finally, the relations (4.55), (4.56) and (4.60) are implied by the block representations (4.80)–(4.82) and the equalities (4.64), (4.65) and (4.71), (4.72).

2) The equalities (4.61) are immediate from (4.21) and the block representations (4.80), (4.81).  $\square$

**Theorem 4.9.** *Let the assumptions of Proposition 4.8 be fulfilled and let  $\tau = \tau(\lambda) \in \widetilde{R}(\mathcal{H}_b)$  be a Nevanlinna operator pair (2.16). Then:*

1) *The statement 1) of Theorem 4.5 holds with the boundary condition (4.37) and the following boundary conditions instead of (4.38)–(4.40):*

$$(4.83) \quad i(P_{\widehat{\mathcal{H}}_b}\widehat{\Gamma}_a - \widehat{\Gamma}_b)v_\tau(\lambda) = P_{\widehat{\mathcal{H}}_b}, \quad \lambda \in \mathbb{C}_+; \quad i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)v_\tau(\lambda) = P_{\widehat{H}}, \quad \lambda \in \mathbb{C}_-$$

$$(4.84) \quad C_0(\lambda)\Gamma_{0b}v_\tau(\lambda) + C_1(\lambda)\Gamma_{1b}v_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

(here  $P_{\widehat{\mathcal{H}}_b}$  ( $P_{\widehat{H}}$ ) is the orthoprojector in  $H_0$  onto  $\widehat{\mathcal{H}}_b$  (resp.  $\widehat{H}$ ) in accordance with the decomposition (3.35));

2) *the operator function  $v_\tau(\cdot, \lambda)$  can be represented as*

$$(4.85) \quad v_\tau(t, \lambda) = v_0(t, \lambda) - u(t, \lambda)(\tau(\lambda) + M_4(\lambda))^{-1}M_3(\lambda)P_{H'_0}, \quad \lambda \in \mathbb{C}_+,$$

$$(4.86) \quad v_\tau(t, \lambda) = v_0(t, \lambda) - u(t, \lambda)(\tau(\lambda) + M_4(\lambda))^{-1}S_-(\lambda), \quad \lambda \in \mathbb{C}_-,$$

where  $v_0(t, \lambda)$  and  $u(t, \lambda)$  are defined in Proposition 4.8 and

$$(4.87) \quad S_-(\lambda) := (M_3(\lambda) : N_{2-}(\lambda)) : H'_0 \oplus \mathcal{H}'_2 \rightarrow \mathcal{H}_b, \quad \lambda \in \mathbb{C}_-.$$

*Proof.* Let us show that the solution  $v_\tau(\cdot, \lambda)$  defined in (4.85) satisfies the boundary conditions (4.37), (4.83) and (4.84). Combining (4.85) with (4.9), (4.54) and (4.57), (4.59) one obtains the relations (4.37) and (4.83). Similarly, the relations (4.55), (4.56) and (4.60) give

$$\Gamma_{0b}v_\tau(\lambda) = -(\tau(\lambda) + M_4(\lambda))^{-1}M_3(\lambda)P_{\mathcal{H}'_0}, \quad \lambda \in \mathbb{C}_+,$$

$$\Gamma_{1b}v_\tau(\lambda) = (-I + M_4(\lambda)(\tau(\lambda) + M_4(\lambda))^{-1})M_3(\lambda)P_{\mathcal{H}'_0}, \quad \lambda \in \mathbb{C}_+,$$

$$\Gamma_{0b}v_\tau(\lambda) = -(\tau(\lambda) + M_4(\lambda))^{-1}S_-(\lambda), \quad \lambda \in \mathbb{C}_-,$$

$$\Gamma_{1b}v_\tau(\lambda) = (-I + M_4(\lambda)(\tau(\lambda) + M_4(\lambda))^{-1})S_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

Hence  $\{\Gamma_{0b}v_\tau(\lambda)h_0, \Gamma_{1b}v_\tau(\lambda)h_0\} \in \tau(\lambda)$ ,  $h_0 \in H_0$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , which yields (4.84). Finally, by using Theorem 4.6 one proves uniqueness of  $v_\tau(\cdot, \lambda)$  in the same way as in Theorem 4.5.  $\square$

**4.3. Case 3.** Application of Corollary 2.15 to the boundary triplet (3.57) gives the following theorem.

**Theorem 4.10.** *Let in Case 3  $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.42), (3.44) for  $T_{max}$ , let  $T$  be a symmetric relation (3.55) and let  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_{-1}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  be a collection of holomorphic operator pairs (2.3). Then for every  $f \in \mathcal{L}_\Delta^2(\mathcal{I})$  the boundary value problem*

$$(4.88) \quad Jy' - B(t)y = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},$$

$$(4.89) \quad \Gamma_{1a}y = 0, \quad C_0(\lambda)\tilde{\Gamma}_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_+,$$

$$(4.90) \quad \Gamma_{1a}y = 0, \quad \hat{\Gamma}_a y = 0, \quad D_0(\lambda)\tilde{\Gamma}_{0b}y + D_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_-.$$

has a unique solution  $y(t, \lambda) = y_f(t, \lambda)$  and the equality (4.4) defines a generalized resolvent  $R(\lambda) =: R_\tau(\lambda)$  of  $T$ . If in addition  $\hat{H} = \{0\}$ , then for each generalized resolvent  $R(\lambda)$  of  $T$  there exists a unique  $\tau \in \tilde{R}_{-1}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  such that  $R(\lambda) = R_\tau(\lambda)$ .

*Remark 4.11.* Let in Theorem 4.10  $\tau_0 = \{\tau_+, \tau_-\} \in \tilde{R}_{-1}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  be defined by (2.3) with

$$(4.91) \quad C_0(\lambda) \equiv P_{\mathcal{H}_b}(\in [\tilde{\mathcal{H}}_b, \mathcal{H}_b]), \quad C_1(\lambda) \equiv 0, \quad D_0(\lambda) \equiv I_{\tilde{\mathcal{H}}_b}, \quad D_1(\lambda) \equiv 0$$

and let  $A_0$  be the symmetric extension (3.58). Then  $R_0(\lambda) := R_{\tau_0}(\lambda)$  is of the form (4.49).

**Proposition 4.12.** *Let in Case 3  $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.42), (3.44) for  $T_{max}$ , let  $\gamma_\pm(\cdot)$  be the  $\gamma$ -fields and let*

$$(4.92) \quad M_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_{2+}(\lambda) \\ N_{1+}(\lambda) & N_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix} : H \oplus \mathcal{H}_b \rightarrow H \oplus \hat{H} \oplus \tilde{\mathcal{H}}_b, \quad \lambda \in \mathbb{C}_+$$

$$(4.93) \quad M_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) & M_{2-}(\lambda) \\ M_{3-}(\lambda) & N_{2-}(\lambda) & M_{4-}(\lambda) \end{pmatrix} : H \oplus \hat{H} \oplus \tilde{\mathcal{H}}_b \rightarrow H \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_-$$

be the block representations of the corresponding Weyl functions. Assume also that  $m_0(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [H_0]$  is the operator function defined by

$$(4.94) \quad m_0(\lambda) = \begin{pmatrix} M_1(\lambda) & 0 \\ N_{1+}(\lambda) & \frac{i}{2}I_{\tilde{\mathcal{H}}_b} \end{pmatrix} : H \oplus \hat{H} \rightarrow H \oplus \hat{H}, \quad \lambda \in \mathbb{C}_+$$

$$(4.95) \quad m_0(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) \\ 0 & -\frac{i}{2}I_{\hat{H}} \end{pmatrix} : H \oplus \hat{H} \rightarrow H \oplus \hat{H}, \quad \lambda \in \mathbb{C}_-.$$

Then: 1) For every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists an operator solution  $v_0(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  of Eq. (3.3) such that (4.9) and (4.10) hold and

$$(4.96) \quad i\hat{\Gamma}_a v_0(\lambda) = P_{\hat{H}}, \quad \lambda \in \mathbb{C}_-,$$

$$(4.97) \quad \tilde{\Gamma}_{0b}v_0(\lambda) = iP_{\tilde{\mathcal{H}}_b}M_{3+}(\lambda)P_H(\in [H_0, \tilde{\mathcal{H}}_b]), \quad \Gamma_{1b}v_0(\lambda) = -P_{\mathcal{H}_b}M_{3+}(\lambda)P_H, \quad \lambda \in \mathbb{C}_+,$$

$$(4.98) \quad \tilde{\Gamma}_{0b}v_0(\lambda) = 0, \quad \Gamma_{1b}v_0(\lambda) = (-M_{3-}(\lambda) : -N_{2-}(\lambda)), \quad \lambda \in \mathbb{C}_-.$$

2) For every  $\lambda \in \mathbb{C}_+$  ( $\lambda \in \mathbb{C}_-$ ) there exists a solution  $u_+(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathcal{H}_b, \mathbb{H}]$  (resp.  $u_-(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\tilde{\mathcal{H}}_b, \mathbb{H}]$ ) of Eq. (3.3) such that

$$(4.99) \quad \Gamma_{1a}u_\pm(\lambda) = 0, \quad \lambda \in \mathbb{C}_\pm,$$

$$(4.100) \quad (\Gamma_{0a} + \hat{\Gamma}_a)u_+(\lambda) = M_{2+}(\lambda) + N_{2+}(\lambda), \quad \lambda \in \mathbb{C}_+,$$

$$(4.101) \quad \Gamma_{0a}u_-(\lambda) = M_{2-}(\lambda), \quad \hat{\Gamma}_au_-(\lambda) = 0, \quad \lambda \in \mathbb{C}_-.$$

$$(4.102) \quad \tilde{\Gamma}_{0b}u_+(\lambda) = I_{\mathcal{H}_b} + iP_{\tilde{\mathcal{H}}_b}M_{4+}(\lambda), \quad \Gamma_{1b}u_+(\lambda) = -P_{\mathcal{H}_b}M_{4+}(\lambda), \quad \lambda \in \mathbb{C}_+,$$

$$(4.103) \quad \tilde{\Gamma}_{0b}u_-(\lambda) = I_{\tilde{\mathcal{H}}_b}, \quad \Gamma_{1b}u_-(\lambda) = -M_{4-}(\lambda), \quad \lambda \in \mathbb{C}_-.$$

3) The following equalities hold

$$(4.104) \quad \gamma_+(\lambda) \upharpoonright \mathcal{H}_b = \pi u_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(\lambda) \upharpoonright \tilde{\mathcal{H}}_b = \pi u_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

*Proof.* As in Proposition 4.4 one proves the existence of isomorphisms  $Z_+(\lambda) : \mathcal{H}_1 \rightarrow \mathcal{N}_\lambda$  ( $\lambda \in \mathbb{C}_+$ ) and  $Z_-(\lambda) : \mathcal{H}_0 \rightarrow \mathcal{N}_\lambda$  ( $\lambda \in \mathbb{C}_-$ ) satisfying (4.62) and (4.63) or, equivalently, the equalities

$$(4.105)$$

$$\begin{pmatrix} -\Gamma_{1a} \\ \Gamma_{0b} \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} I_H & 0 \\ 0 & I_{\mathcal{H}_b} \end{pmatrix}, \quad \begin{pmatrix} \Gamma_{0a} \\ \hat{\Gamma}_a \\ -\Gamma_{1b} - i\hat{\Gamma}_b \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_{2+}(\lambda) \\ N_{1+}(\lambda) & N_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_+,$$

$$(4.106) \quad \begin{pmatrix} -\Gamma_{1a} \\ i\hat{\Gamma}_a \\ \tilde{\Gamma}_{0b} \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} I_H & 0 & 0 \\ 0 & I_{\hat{H}} & 0 \\ 0 & 0 & I_{\tilde{\mathcal{H}}_b} \end{pmatrix}, \quad \lambda \in \mathbb{C}_-$$

$$(4.107) \quad \begin{pmatrix} \Gamma_{0a} \\ -\Gamma_{1b} \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) & M_{2-}(\lambda) \\ M_{3-}(\lambda) & N_{2-}(\lambda) & M_{4-}(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_-.$$

It follows from (4.105)–(4.107) that

$$(4.108) \quad \Gamma_{1a}Z_+(\lambda) = (-I_H : 0), \quad \lambda \in \mathbb{C}_+; \quad \Gamma_{1a}Z_-(\lambda) = (-I_H : 0 : 0), \quad \lambda \in \mathbb{C}_-,$$

$$(4.109) \quad (\Gamma_{0a} + \hat{\Gamma}_a)Z_+(\lambda) = \begin{pmatrix} \Gamma_{0a} \\ \hat{\Gamma}_a \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_{2+}(\lambda) \\ N_{1+}(\lambda) & N_{2+}(\lambda) \end{pmatrix} : H \oplus \mathcal{H}_b \rightarrow H \oplus \hat{H},$$

$$(4.110) \quad (\Gamma_{0a} + \hat{\Gamma}_a)Z_-(\lambda) = \begin{pmatrix} \Gamma_{0a} \\ \hat{\Gamma}_a \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) & M_{2-}(\lambda) \\ 0 & -iI_{\hat{H}} & 0 \end{pmatrix} : H \oplus \hat{H} \oplus \tilde{\mathcal{H}}_b \rightarrow H \oplus \hat{H},$$

$$(4.111) \quad \Gamma_{0b}Z_+(\lambda) = (0 : I_{\mathcal{H}_b}), \quad \hat{\Gamma}_bZ_+(\lambda) = (iP_{\tilde{\mathcal{H}}_b}M_{3+}(\lambda) : iP_{\tilde{\mathcal{H}}_b}M_{4+}(\lambda)), \quad \lambda \in \mathbb{C}_+,$$

$$(4.112) \quad \Gamma_{1b}Z_+(\lambda) = (-P_{\mathcal{H}_b}M_{3+}(\lambda) : -P_{\mathcal{H}_b}M_{4+}(\lambda)), \quad \lambda \in \mathbb{C}_+,$$

$$(4.113)$$

$$\tilde{\Gamma}_{0b}Z_-(\lambda) = (0 : 0 : I_{\tilde{\mathcal{H}}_b}), \quad \Gamma_{1b}Z_-(\lambda) = -(M_{3-}(\lambda) : N_{2-}(\lambda) : M_{4-}(\lambda)), \quad \lambda \in \mathbb{C}_-,$$

and combining of (3.40) with (4.111) gives

$$(4.114) \quad \tilde{\Gamma}_{0b}Z_+(\lambda) = (iP_{\tilde{\mathcal{H}}_b}M_{3+}(\lambda) : I_{\mathcal{H}_b} + iP_{\tilde{\mathcal{H}}_b}M_{4+}(\lambda)), \quad \lambda \in \mathbb{C}_+.$$

Assume that

$$(4.115) \quad Z_+(\lambda) = (r(\lambda) : u_+(\lambda)) : H \oplus \mathcal{H}_b \rightarrow \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_+,$$

$$(4.116) \quad Z_-(\lambda) = (v_0(\lambda) : u_-(\lambda)) : H_0 \oplus \tilde{\mathcal{H}}_b \rightarrow \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_-,$$

are the block representations of  $Z_\pm(\lambda)$  (see (3.41)) and let

$$(4.117) \quad v_0(\lambda) := (r(\lambda) : 0) : H \oplus \hat{H} \rightarrow \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_+.$$

Then the equalities (4.115)–(4.117) define the linear mappings  $v_0(\lambda) : H_0 \rightarrow \mathcal{N}_\lambda$  ( $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ),  $u_+(\lambda) : \mathcal{H}_b \rightarrow \mathcal{N}_\lambda$  ( $\lambda \in \mathbb{C}_+$ ) and  $u_-(\lambda) : \tilde{\mathcal{H}}_b \rightarrow \mathcal{N}_\lambda$  ( $\lambda \in \mathbb{C}_-$ ), which according to Lemma 3.2 generate the operator solutions  $v_0(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$ ,  $u_+(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathcal{H}_b, \mathbb{H}]$  and  $u_-(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\tilde{\mathcal{H}}_b, \mathbb{H}]$  of Eq. (3.3). Combining now (4.115)–(4.117) with (4.108)–(4.110) and taking the equalities (4.94) and (4.95) into account we arrive at the relations (4.9), (4.10), (4.96) and (4.99)–(4.101). Moreover, the block representations (4.115)–(4.117) and the equalities (4.112)–(4.114) lead to (4.97), (4.98) and (4.102), (4.103).

Finally, (4.104) is implied by (4.21) and (4.115), (4.116).  $\square$

**Theorem 4.13.** *Let the assumptions of Proposition 4.12 be satisfied, let  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_{-1}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  be a collection of operator pairs (2.3) and let*

$$(4.118) \quad S_-(\lambda) = (M_{3-}(\lambda) : N_{2-}(\lambda)) : H \oplus \hat{H} \rightarrow \mathcal{H}_b, \quad \lambda \in \mathbb{C}_-.$$

*Then: 1) the statement 1) of Theorem 4.5 holds with the boundary condition (4.37) and the following boundary conditions in place of (4.38)–(4.40):*

$$(4.119) \quad i\hat{\Gamma}_a v_\tau(\lambda) = P_{\hat{H}}, \quad \lambda \in \mathbb{C}_-,$$

$$(4.120) \quad C_0(\lambda)\tilde{\Gamma}_{0b}v_\tau(\lambda) + C_1(\lambda)\Gamma_{1b}v_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C}_+,$$

$$(4.121) \quad D_0(\lambda)\tilde{\Gamma}_{0b}v_\tau(\lambda) + D_1(\lambda)\Gamma_{1b}v_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C}_-.$$

*2) the solution  $v_\tau(\cdot, \lambda)$  is of the form*

$$(4.122) \quad v_\tau(t, \lambda) = v_0(t, \lambda) - u_+(t, \lambda)(\tau_-^*(\bar{\lambda}) + M_{4+}(\lambda))^{-1}M_{3+}(\lambda)P_H, \quad \lambda \in \mathbb{C}_+$$

$$(4.123) \quad v_\tau(t, \lambda) = v_0(t, \lambda) - u_-(t, \lambda)(\tau_-(\lambda) + M_{4-}(\lambda))^{-1}S_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

*Proof.* Let us show that the solution  $v_\tau(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  of Eq. (3.3) defined by (4.122) and (4.123) satisfies the boundary conditions (4.37) and (4.119)–(4.121).

Combining (4.122) and (4.123) with (4.9) and (4.99) we obtain the equality (4.37). Moreover, (4.122) and (4.123) together with (4.96) and the second equality in (4.101) give (4.119).

Next assume that  $T_+(\lambda) = (\tau_-^*(\bar{\lambda}) + M_{4+}(\lambda))^{-1}$ ,  $\lambda \in \mathbb{C}_+$ , and  $T_-(\lambda) = (\tau_-(\lambda) + M_{4-}(\lambda))^{-1}$ ,  $\lambda \in \mathbb{C}_-$ . Then by using (2.11) one obtains

$$(4.124) \quad \tau_+(\lambda) = \{ \{ (-T_+(\lambda) + iP_{\hat{\mathcal{H}}_b} - iP_{\hat{\mathcal{H}}_b} M_{4+}(\lambda) T_+(\lambda)) h, \\ (-P_{\mathcal{H}_b} + P_{\mathcal{H}_b} M_{4+}(\lambda) T_+(\lambda)) h \} : h \in \tilde{\mathcal{H}}_b \},$$

$$(4.125) \quad \tau_-(\lambda) = \{ \{ T_-(\lambda) h, (I - M_{4-}(\lambda) T_-(\lambda)) h \} : h \in \mathcal{H}_b \}$$

(c.f. proofs of the equalities (4.43) and (4.44) in Theorem 4.5). Moreover, combining (4.122) and (4.123) with the equalities (4.97), (4.98) and (4.102), (4.103) one gets

$$\begin{aligned}\tilde{\Gamma}_{0b}v_\tau(\lambda) &= (iP_{\tilde{\mathcal{H}}_b} - T_+(\lambda) - iP_{\tilde{\mathcal{H}}_b}M_{4+}(\lambda)T_+(\lambda))M_{3+}(\lambda)P_H, \quad \lambda \in \mathbb{C}_+, \\ \Gamma_{1b}v_\tau(\lambda) &= (-P_{\mathcal{H}_b} + P_{\mathcal{H}_b}M_{4+}(\lambda)T_+(\lambda))M_{3+}(\lambda)P_H, \quad \lambda \in \mathbb{C}_+, \\ \tilde{\Gamma}_{0b}v_\tau(\lambda) &= -T_-(\lambda)S_-(\lambda), \quad \Gamma_{1b}v_\tau(\lambda) = -(I - M_{4-}(\lambda)T_-(\lambda))S_-(\lambda), \quad \lambda \in \mathbb{C}_-, \end{aligned}$$

which in view of (4.124) and (4.125) yields the inclusions (4.45). This implies that  $v_\tau(\cdot, \lambda)$  satisfies (4.120) and (4.121). Finally, by using the same arguments as in Theorem 4.5 one proves uniqueness of the solution  $v_\tau(\cdot, \lambda)$ .  $\square$

## 5. $m$ -FUNCTIONS

Assume that  $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  is a decomposing boundary triplet for  $T_{max}$  defined by (3.44) and one of the equalities (3.31), (3.38) or (3.42).

**Definition 5.1.** A boundary parameter  $\tau$  (at the endpoint  $b$ ) is:

- a collection  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_\alpha(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  of operator pairs (2.3) with  $\alpha = +1$  in *Case 1* and  $\alpha = -1$  in *Case 3*;
- an operator pair  $\tau \in \tilde{R}(\mathcal{H}_b)$  defined by (2.16) in *Case 2*.

If  $n_+ = n_-$  and  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  is a decomposing boundary triplet (3.34), (3.44) for  $T_{max}$ , then a boundary parameter is an operator pair  $\tau = \tilde{R}(\mathcal{H}_b)$  of the form (2.16).

Let  $\tau$  be a boundary parameter and let  $v_\tau(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  be the corresponding operator solution of Eq. (3.3) defined in Theorems 4.5, 4.9 and 4.13.

**Definition 5.2.** The operator function  $m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [H_0]$  defined by

$$(5.1) \quad m_\tau(\lambda) = (\Gamma_{0a} + \hat{\Gamma}_a)v_\tau(\lambda) + \frac{i}{2}P_{\hat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

will be called the  $m$ -function corresponding to the boundary parameter  $\tau$  or, equivalently, to the boundary value problem (4.1)–(4.3) (in *Case 1*), (4.46)–(4.48) (in *Case 2*) or (4.88)–(4.90) (in *Case 3*).

If  $n_+ = n_-$ , then  $m_\tau(\cdot)$  corresponds to the boundary value problem (4.1), (4.5). In this case the  $m$ -function  $m_\tau(\cdot)$  will be called canonical if  $\tau \in \tilde{R}^0(\mathcal{H}_b)$ .

It follows from (4.37) that  $m_\tau(\cdot)$  satisfies the equality

$$(5.2) \quad v_{\tau,a}(\lambda) \begin{pmatrix} \Gamma_{0a} + \hat{\Gamma}_a \\ \Gamma_{1a} \end{pmatrix} v_\tau(\lambda) = \begin{pmatrix} m_\tau(\lambda) - \frac{i}{2}P_{\hat{H}} \\ -P_H \end{pmatrix} : H_0 \rightarrow H_0 \oplus H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In the following proposition we show that the  $m$ -function  $m_\tau(\cdot)$  can be defined in a somewhat different way.

**Proposition 5.3.** Let  $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet for  $T_{max}$ , let  $\tau$  be a boundary parameter at the endpoint  $b$  and let  $\varphi(\cdot, \lambda) (\in [H_0, \mathbb{H}])$  and  $\psi(\cdot, \lambda) (\in [H_0, \mathbb{H}])$ ,  $\lambda \in \mathbb{C}$ , be the operator solutions of Eq. (3.3) with the initial data

$$(5.3) \quad \varphi_a(\lambda) = \begin{pmatrix} I_{H_0} \\ 0 \end{pmatrix} : H_0 \rightarrow H_0 \oplus H, \quad \psi_a(\lambda) = \begin{pmatrix} -\frac{i}{2}P_{\hat{H}} \\ -P_H \end{pmatrix} : H_0 \rightarrow H_0 \oplus H, \quad \lambda \in \mathbb{C}.$$

Then there exists a unique operator function  $m(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [H_0]$  such that for every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  the operator solution  $v(\cdot, \lambda)$  of Eq. (3.3) given by

$$(5.4) \quad v(t, \lambda) := \varphi(t, \lambda)m(\lambda) + \psi(t, \lambda)$$

belongs to  $\mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  and satisfies the following boundary conditions: (4.38)–(4.40) in Case 1; (4.83) and (4.84) in Case 2; (4.119)–(4.121) in Case 3. Moreover, the equalities  $v(t, \lambda) = v_\tau(t, \lambda)$  and  $m(\lambda) = m_\tau(\lambda)$  are valid.

*Proof.* Let  $m_\tau(\cdot)$  be the  $m$ -function in the sense of Definition 5.2 and let  $v(\cdot, \lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , be the solution of Eq. (3.3) given by (5.4) with  $m(\lambda) = m_\tau(\lambda)$ . Then in view of (5.3) and (5.2) one has  $v_a(\lambda) = v_{\tau,a}(\lambda)$  and, consequently,  $v(t, \lambda) = v_\tau(t, \lambda)$ . Therefore by Theorems 4.5, 4.9 and 4.13  $v(\cdot, \lambda)$  belongs to  $\mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  and satisfies the required boundary conditions. Hence there exists an operator function  $m(\lambda) (= m_\tau(\lambda))$  with the desired properties.

Assume now that the solution  $v(\cdot, \lambda)$  of Eq. (3.3) given by (5.4) with some  $m(\lambda)$  belongs to  $\mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  and satisfies the specified boundary conditions. Then in view of (5.3)  $\Gamma_{1a}v(\lambda) = -P_H$  and according to Theorems 4.5, 4.9 and 4.13  $v(t, \lambda) = v_\tau(t, \lambda)$ . Therefore  $m(\lambda) = m_\tau(\lambda)$ , which proves uniqueness of  $m(\lambda)$ .  $\square$

Description of all  $m$ -functions immediately in terms of the boundary parameter  $\tau$  is contained in the following three theorems.

**Theorem 5.4.** *Let in Case 1 the assumptions of Proposition 4.4 be satisfied and let  $\tau_0 = \{\tau_+, \tau_-\}$  be a boundary parameter (2.3), (4.6). Then  $m_0(\lambda) = m_{\tau_0}(\lambda)$  and for every boundary parameter  $\tau = \{\tau_+, \tau_-\}$  defined by (2.3) the corresponding  $m$ -function  $m_\tau(\cdot)$  is of the form*

$$(5.5) \quad m_\tau(\lambda) = m_0(\lambda) + M_{2+}(\lambda)(C_0(\lambda) - C_1(\lambda)M_{4+}(\lambda))^{-1}C_1(\lambda)M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+.$$

*Proof.* It follows from (4.9) and (4.11)–(4.13) that  $v_0(t, \lambda) = v_{\tau_0}(t, \lambda)$  and (4.10) yields  $m_0(\lambda) = m_{\tau_0}(\lambda)$ . Next, applying the operator  $\Gamma_{0a} + \widehat{\Gamma}_a$  to the equalities (4.41) and (4.42) with taking (4.10) and (4.15) into account one obtains

$$(5.6) \quad m_\tau(\lambda) = m_0(\lambda) - M_{2+}(\lambda)(\tau_+(\lambda) + M_{4+}(\lambda))^{-1}M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+,$$

$$(5.7) \quad m_\tau(\lambda) = m_0(\lambda) - M_{2-}(\lambda)(\tau_+^*(\bar{\lambda}) + M_{4-}(\lambda))^{-1}M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-.$$

It follows from the equality  $M_+^*(\bar{\lambda}) = M_-(\lambda)$  that  $m_0^*(\bar{\lambda}) = m_0(\lambda)$ ,  $M_{2+}^*(\bar{\lambda}) = M_{3-}(\lambda)$ ,  $M_{3+}^*(\bar{\lambda}) = M_{2-}(\lambda)$  and  $M_{4+}^*(\bar{\lambda}) = M_{4-}(\lambda)$ . This and (5.6), (5.7) yield

$$(5.8) \quad m_\tau^*(\bar{\lambda}) = m_\tau(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, according to [31, Lemma 2.1]  $0 \in \rho(C_0(\lambda) - C_1(\lambda)M_{4+}(\lambda))$  and

$$-(\tau_+(\lambda) + M_{4+}(\lambda))^{-1} = (C_0(\lambda) - C_1(\lambda)M_{4+}(\lambda))^{-1}C_1(\lambda), \quad \lambda \in \mathbb{C}_+,$$

which together with (5.6) yields (5.5).  $\square$

The following corollary is immediate from Theorem 5.4 and the equalities (3.32).

**Corollary 5.5.** *Let  $n_+ = n_-$ , let  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.34), (3.44) for  $T_{max}$ , let  $\tau_0 = \{(I_{\mathcal{H}_b}, 0); \mathcal{H}_b\} \in \widehat{R}^0(\mathcal{H}_b)$  and let*

$$(5.9) \quad M(\lambda) = \begin{pmatrix} m_0(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : H_0 \oplus \mathcal{H}_b \rightarrow H_0 \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

*be the block representation of the Weyl function of  $\Pi$ . Then  $m_0(\lambda) = m_{\tau_0}(\lambda)$  and for every boundary parameter  $\tau$  defined by (2.16) the corresponding  $m$ -function  $m_\tau(\cdot)$  is*

$$(5.10) \quad m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$



**Theorem 5.6.** *Let in Case 2 the assumptions of Proposition 4.8 be satisfied, let  $\tau_0 = \{(I_{\mathcal{H}_b}, 0); \mathcal{H}_b\} \in \tilde{R}^0(\mathcal{H}_b)$  and let  $S(\lambda)$  be the operator function (4.87).*

*Then: 1)  $m_0(\lambda) = m_{\tau_0}(\lambda)$  and for every boundary parameter  $\tau$  defined by (2.16) the corresponding  $m$ -function  $m_\tau(\cdot)$  is*

$$(5.11) \quad m_\tau(\lambda) = m_0(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)S_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

*2)  $m_\tau(\lambda)$  admits the triangular block representation*

$$(5.12) \quad m_\tau(\lambda) = \begin{pmatrix} m_{1,\tau}(\lambda) & 0 \\ m_{+,\tau}(\lambda) & \frac{i}{2}I_{\mathcal{H}'_2} \end{pmatrix} : \underbrace{H'_0 \oplus \mathcal{H}'_2}_{H_0} \rightarrow \underbrace{H'_0 \oplus \mathcal{H}'_2}_{H_0}, \quad \lambda \in \mathbb{C}_+$$

$$(5.13) \quad m_\tau(\lambda) = \begin{pmatrix} m_{1,\tau}(\lambda) & m_{-,\tau}(\lambda) \\ 0 & -\frac{i}{2}I_{\mathcal{H}'_2} \end{pmatrix} : \underbrace{H'_0 \oplus \mathcal{H}'_2}_{H_0} \rightarrow \underbrace{H'_0 \oplus \mathcal{H}'_2}_{H_0}, \quad \lambda \in \mathbb{C}_-,$$

where

$$(5.14) \quad m_{1,\tau}(\lambda) = M_1(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(5.15) \quad m_{-,\tau}(\lambda) = N_{1-}(\lambda) + M_2(\lambda)(C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)N_{2-}(\lambda), \quad \lambda \in \mathbb{C}_-,$$

$$(5.16) \quad m_{+,\tau}(\lambda) = m_{-,\tau}^*(\bar{\lambda}), \quad \lambda \in \mathbb{C}_+.$$

*Proof.* 1) The equality  $m_0(\lambda) = m_{\tau_0}(\lambda)$  is implied by (4.9), (4.54) and (4.55) in the same way as in Theorem 5.4. Next, by (4.85), (4.86) and (4.10), (4.58) one has

$$(5.17) \quad m_\tau(\lambda) = m_0(\lambda) - (M_2(\lambda) + N_{2+}(\lambda))(\tau(\lambda) + M_4(\lambda))^{-1}M_3(\lambda)P_{H'_0}, \quad \lambda \in \mathbb{C}_+,$$

$$(5.18) \quad m_\tau(\lambda) = m_0(\lambda) - M_2(\lambda)(\tau(\lambda) + M_4(\lambda))^{-1}M_3(\lambda)S_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

It follows from (4.50), (4.51) and the equality  $M_+^*(\bar{\lambda}) = M_-(\lambda)$  that for all  $\lambda \in \mathbb{C}_-$

$$\begin{aligned} (M_3(\bar{\lambda})P_{H'_0})^* &= M_2(\lambda), & M_4^*(\bar{\lambda}) &= M_4(\lambda), \\ (M_2(\bar{\lambda}) + N_{2+}(\bar{\lambda}))^* &= (M_3(\lambda) : N_{2-}(\lambda)) = S_-(\lambda). \end{aligned}$$

This and (5.17), (5.18) yield the equality (5.8). Moreover, applying to (5.18) the same arguments as in the proof of Theorem 5.4 one obtains (5.11).

2) It follows from (5.11) and (4.87) that

$$m_\tau(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) \\ 0 & -\frac{i}{2}I_{\mathcal{H}'_2} \end{pmatrix} + \begin{pmatrix} M_2(\lambda) \\ 0 \end{pmatrix} (C_0(\lambda) - C_1(\lambda)M_4(\lambda))^{-1}C_1(\lambda)(M_3(\lambda) : N_{2-}(\lambda))$$

for all  $\lambda \in \mathbb{C}_-$ . This proves (5.13)–(5.15), which in view of (5.8) implies (5.12) and (5.16).  $\square$

**Theorem 5.7.** *Let in Case 3 the conditions of Proposition 4.12 be fulfilled, let  $\tau_0 = \{\tau_+, \tau_-\}$  be a boundary parameter (2.3), (4.91) and let  $S_-(\lambda)$  be the operator function (4.118). Then:*

*1)  $m_0(\lambda) = m_{\tau_0}(\lambda)$  and for every boundary parameter  $\tau = \{\tau_+, \tau_-\}$  defined by (2.3) the  $m$ -function  $m_\tau(\cdot)$  is of the form*

$$(5.19) \quad m_\tau(\lambda) = m_0(\lambda) + M_{2-}(\lambda)(D_0(\lambda) - D_1(\lambda)M_{4-}(\lambda))^{-1}D_1(\lambda)S_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

2) The  $m$ -function  $m_\tau(\cdot)$  has the triangular block representation

$$(5.20) \quad m_\tau(\lambda) = \begin{pmatrix} m_{1,\tau}(\lambda) & 0 \\ m_{+,\tau}(\lambda) & \frac{i}{2}I_{\widehat{H}} \end{pmatrix} : \underbrace{H \oplus \widehat{H}}_{H_0} \rightarrow \underbrace{H \oplus \widehat{H}}_{H_0}, \quad \lambda \in \mathbb{C}_+$$

$$(5.21) \quad m_\tau(\lambda) = \begin{pmatrix} m_{1,\tau}(\lambda) & m_{-,\tau}(\lambda) \\ 0 & -\frac{i}{2}I_{\widehat{H}} \end{pmatrix} : \underbrace{H \oplus \widehat{H}}_{H_0} \rightarrow \underbrace{H \oplus \widehat{H}}_{H_0}, \quad \lambda \in \mathbb{C}_-,$$

where

$$(5.22) \quad m_{1,\tau}(\lambda) = M_1(\lambda) + M_{2-}(\lambda)(D_0(\lambda) - D_1(\lambda)M_{4-}(\lambda))^{-1}D_1(\lambda)M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-,$$

$$(5.23) \quad m_{-,\tau}(\lambda) = N_{1-}(\lambda) + M_{2-}(\lambda)(D_0(\lambda) - D_1(\lambda)M_{4-}(\lambda))^{-1}D_1(\lambda)N_{2-}(\lambda), \quad \lambda \in \mathbb{C}_-,$$

$$(5.24) \quad m_{1,\tau}(\lambda) = m_{1,\tau}^*(\bar{\lambda}), \quad m_{+,\tau}(\lambda) = m_{+,\tau}^*(\bar{\lambda}), \quad \lambda \in \mathbb{C}_+.$$

*Proof.* We give only the sketch of the proof, because it is similar to that of Theorems 5.4 and 5.6. The equality  $m_0(\lambda) = m_{\tau_0}(\lambda)$  follows from (4.9) and (4.96)-(4.98). Next, by using (4.10), (4.100), (4.101) and (4.122),(4.123) one proves the equalities

$$m_\tau(\lambda) = m_0(\lambda) - (M_{2+}(\lambda) + N_{2+}(\lambda))(\tau_-^*(\bar{\lambda}) + M_{4+}(\lambda))^{-1}M_{3+}(\lambda)P_H, \quad \lambda \in \mathbb{C}_+,$$

$$m_\tau(\lambda) = m_0(\lambda) - M_{2-}(\lambda)(\tau_-(\lambda) + M_{4-}(\lambda))^{-1}S_-(\lambda), \quad \lambda \in \mathbb{C}_-,$$

which imply (5.8) and (5.19). Moreover, in view of (4.118) the equality (5.19) can be written as

$$m_\tau(\lambda) = \begin{pmatrix} M_1(\lambda) & N_{1-}(\lambda) \\ 0 & -\frac{i}{2}I_{\widehat{H}} \end{pmatrix} + \begin{pmatrix} M_{2-}(\lambda) \\ 0 \end{pmatrix} (D_0(\lambda) - D_1(\lambda)M_{4-}(\lambda))^{-1}D_1(\lambda)(M_{3-}(\lambda) : N_{2-}(\lambda)).$$

This and (5.8) yield (5.20)-(5.24).  $\square$

**Proposition 5.8.** *The  $m$ -function  $m_\tau(\cdot)$  is a Nevanlinna operator function such that the relation*

$$(5.25) \quad (\operatorname{Im} \lambda)^{-1} \cdot \operatorname{Im} m_\tau(\lambda) \geq \int_{\mathcal{I}} v_\tau^*(t, \lambda) \Delta(t) v_\tau(t, \lambda) dt$$

holds for all  $\lambda \in \mathbb{C}_+$  in Case 1 and  $\lambda \in \mathbb{C}_-$  in Cases 2 and 3.

If in addition  $n_+ = n_-$ , then (5.25) holds for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

*Proof.* We prove the proposition only for Case 1 (in Cases 2 and 3 the proof is similar).

Let  $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (3.44), (3.31) for  $T_{max}$  and let  $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}_{+1}(\widetilde{\mathcal{H}}_b, \mathcal{H}_b)$  be a boundary parameter defined by (2.3). Let us show that the corresponding  $m$ -function  $m_\tau(\cdot)$  satisfies (5.25).

Assume that  $\lambda \in \mathbb{C}_+$ ,  $h_0 \in H_0$  and let  $y := v_\tau(\lambda)h_0$ , so that  $y = y(t) = v_\tau(t, \lambda)h_0$ ,  $t \in \mathcal{I}$ . Applying the Lagrange's identity (3.5) to  $\{y, \lambda y\} \in \mathcal{T}_{max}$  and taking the equalities (1.3) and (3.16) into account one obtains

$$(5.26) \quad \operatorname{Im} \lambda \cdot (y, y)_\Delta = \frac{1}{2}(\|\widehat{\Gamma}_b y\|^2 - \|\widehat{\Gamma}_a y\|^2) + \operatorname{Im}(\Gamma_{1a}y, \Gamma_{0a}y) - \operatorname{Im}(\Gamma_{1b}y, \Gamma_{0b}y).$$

It follows from (4.38) that  $P_{\widehat{H}}\widehat{\Gamma}_b y = \widehat{\Gamma}_a y + iP_{\widehat{H}}h_0$  and, therefore,

$$(5.27) \quad \|P_{\widehat{H}}\widehat{\Gamma}_b y\|^2 = \|\widehat{\Gamma}_a y\|^2 + \|P_{\widehat{H}}h_0\|^2 + 2\operatorname{Im}(\widehat{\Gamma}_a y, P_{\widehat{H}}h_0).$$

Moreover, in view of (3.29) one has

$$(5.28) \quad P_{\mathcal{H}'_2}\widehat{\Gamma}_b y = P_{\mathcal{H}'_2}\widetilde{\Gamma}_{0b}y, \quad (\Gamma_{1b}y, \Gamma_{0b}y) = (\Gamma_{1b}y, \widetilde{\Gamma}_{0b}y).$$

Now by using first the decomposition (3.28) and then the equality (5.27) one gets

$$(5.29) \quad \begin{aligned} \|\widehat{\Gamma}_b y\|^2 - \|\widehat{\Gamma}_a y\|^2 &= \|P_{\widehat{H}} \widehat{\Gamma}_b y\|^2 + \|P_{\mathcal{H}'_2} \widehat{\Gamma}_b y\|^2 - \|\widehat{\Gamma}_a y\|^2 = \\ &= \|P_{\widehat{H}} h_0\|^2 + 2\operatorname{Im}(\widehat{\Gamma}_a y, P_{\widehat{H}} h_0) + \|P_{\mathcal{H}'_2} \widetilde{\Gamma}_{0b} y\|^2. \end{aligned}$$

Next, according to (5.2)

$$(5.30) \quad \Gamma_{0a} y = P_H m_\tau(\lambda) h_0, \quad \Gamma_{1a} y = -P_H h_0,$$

$$(5.31) \quad \widehat{\Gamma}_a y = P_{\widehat{H}} m_\tau(\lambda) h_0 - \frac{i}{2} P_{\widehat{H}} h_0$$

and substitution of (5.31) to (5.29) yields

$$(5.32) \quad \|\widehat{\Gamma}_b y\|^2 - \|\widehat{\Gamma}_a y\|^2 = 2\operatorname{Im}(P_{\widehat{H}} m_\tau(\lambda) h_0, P_{\widehat{H}} h_0) + \|P_{\mathcal{H}'_2} \widetilde{\Gamma}_{0b} y\|^2.$$

Moreover, by (5.30) one has

$$(5.33) \quad \operatorname{Im}(\Gamma_{1a} y, \Gamma_{0a} y) = \operatorname{Im}(P_H m_\tau(\lambda) h_0, P_H h_0).$$

Substituting now (5.32) and (5.33) to (5.26) and taking the second equality in (5.28) into account we obtain

$$(5.34) \quad \operatorname{Im} \lambda \cdot (y, y)_\Delta = \operatorname{Im}(m_\tau(\lambda) h_0, h_0) - (\operatorname{Im}(\Gamma_{1b} y, \widetilde{\Gamma}_{0b} y) - \frac{1}{2} \|P_{\mathcal{H}'_2} \widetilde{\Gamma}_{0b} y\|^2).$$

It follows from (4.39) that  $\{\widetilde{\Gamma}_{0b} y, \Gamma_{1b} y\} \in \tau_+(\lambda)$ . Therefore according to [32, Proposition 4.3]

$$(5.35) \quad \operatorname{Im}(\Gamma_{1b} y, \widetilde{\Gamma}_{0b} y) - \frac{1}{2} \|P_{\mathcal{H}'_2} \widetilde{\Gamma}_{0b} y\|^2 \geq 0.$$

Moreover, in view of (3.1) one has

$$(5.36) \quad (y, y)_\Delta = \int_{\mathcal{I}} (\Delta(t) v_\tau(t, \lambda) h_0, v_\tau(t, \lambda) h_0) dt = ((\int_{\mathcal{I}} v_\tau^*(t, \lambda) \Delta(t) v_\tau(t, \lambda) dt) h_0, h_0).$$

Combining now (5.35) and (5.36) with (5.34) we arrive at the relation (5.25).

It follows from (5.5) that the operator function  $m_\tau(\cdot)$  is holomorphic in  $\mathbb{C}_+$ . Moreover, the relation (5.25) shows that  $\operatorname{Im} m_\tau(\lambda) \geq 0$  for all  $\lambda \in \mathbb{C}_+$ . This and (5.8) imply that  $m_\tau(\cdot)$  is a Nevanlinna function.  $\square$

In the following proposition we show that a canonical  $m$ -function  $m_\tau(\cdot)$  is the Weyl function of some symmetric extension of  $T_{\min}$ .

**Proposition 5.9.** *Assume that  $n_+ = n_-$  and  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  is a decomposing boundary triplet (3.34), (3.44) for  $T_{\max}$ . Moreover, let  $\tau \in \widetilde{R}^0(\mathcal{H}_b)$  be a boundary parameter (2.19) and let  $v_\tau(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  be the operator solution of Eq. (3.3) defined in Theorem 4.5.*

*Then: 1) The equalities*

$$\begin{aligned} \widetilde{T} &= \{\{\widetilde{y}, \widetilde{f}\} \in T_{\max} : y(a) = 0, \widehat{\Gamma}_a y = \widehat{\Gamma}_b y, C_0 \Gamma_{0b} y + C_1 \Gamma_{1b} y = 0\}, \\ \widetilde{T}^* &= \{\{\widetilde{y}, \widetilde{f}\} \in T_{\max} : C_0 \Gamma_{0b} y + C_1 \Gamma_{1b} y = 0\} \end{aligned}$$

*define a symmetric extension  $\widetilde{T}$  of  $T_{\min}$  and its adjoint  $\widetilde{T}^*$ ;*

*2) The collection  $\Pi^n = \{H_0, \Gamma_0^n, \Gamma_1^n\}$  with the operators*

$$(5.37) \quad \Gamma_0^n \{\widetilde{y}, \widetilde{f}\} = -\Gamma_{1a} y + i(\widehat{\Gamma}_a - \widehat{\Gamma}_b) y, \quad \Gamma_1^n \{\widetilde{y}, \widetilde{f}\} = \Gamma_{0a} y + \frac{1}{2}(\widehat{\Gamma}_a + \widehat{\Gamma}_b) y, \quad \{\widetilde{y}, \widetilde{f}\} \in \widetilde{T}^*,$$

*is a boundary triplet for  $\widetilde{T}^*$ . Moreover, the  $\gamma$ -field  $\gamma^n(\cdot)$  and Weyl function  $M^n(\cdot)$  of  $\Pi^n$  are*

$$(5.38) \quad \gamma^n(\lambda) = \pi v_\tau(\lambda), \quad M^n(\lambda) = m_\tau(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R};$$

3) the following identity holds

$$(5.39) \quad m_\tau(\mu) - m_\tau^*(\lambda) = (\mu - \bar{\lambda}) \int_{\mathcal{I}} v_\tau^*(t, \lambda) \Delta(t) v_\tau(t, \mu) dt, \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

This implies that for the canonical  $m$ -function  $m_\tau(\cdot)$  the inequality (5.25) turns into the equality, which holds for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .

*Proof.* Clearly, we may assume that  $\tau$  is given in the normalized form (2.20), in which case the operators

$$(5.40) \quad \bar{\Gamma}_0\{\tilde{y}, \tilde{f}\} = \{-\Gamma_{1a}y + i(\hat{\Gamma}_a - \hat{\Gamma}_b)y, C_0\Gamma_{0b}y + C_1\Gamma_{1b}y\}(\in H_0 \oplus \mathcal{H}_b),$$

$$(5.41) \quad \bar{\Gamma}_1\{\tilde{y}, \tilde{f}\} = \{\Gamma_{0a}y + \frac{1}{2}(\hat{\Gamma}_a + \hat{\Gamma}_b)y, C_1\Gamma_{0b}y - C_0\Gamma_{1b}y\}(\in H_0 \oplus \mathcal{H}_b), \quad \{\tilde{y}, \tilde{f}\} \in T_{max},$$

form a decomposing boundary triplet  $\bar{\Pi} = \{\mathcal{H}, \bar{\Gamma}_0, \bar{\Gamma}_1\}$  for  $T_{max}$ . Let  $\bar{\gamma}(\lambda)$  be the  $\gamma$ -field and

$$(5.42) \quad \bar{M}(\lambda) = \begin{pmatrix} \bar{m}_0(\lambda) & \bar{M}_2(\lambda) \\ \bar{M}_3(\lambda) & \bar{M}_4(\lambda) \end{pmatrix} : H_0 \oplus \mathcal{H}_b \rightarrow H_0 \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

be the Weyl function of the triplet  $\bar{\Pi}$ . Assume also that  $\bar{v}_0(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  is the operator solution of Eq. (3.3) defined in Proposition 4.4 (for the triplet  $\bar{\Pi}$ ). Then  $\bar{v}_0(t, \lambda) = v_\tau(t, \lambda)$  and (4.19) yields  $\bar{\gamma}(\lambda) \upharpoonright H_0 = \pi v_\tau(\lambda)$ . Moreover, in view of (4.10) one has  $\bar{m}_0(\lambda) = m_\tau(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Applying now Proposition 2.12 to the triplet  $\bar{\Pi}$  (with  $\dot{\mathcal{H}}_0 = \dot{\mathcal{H}}_1 = H_0$ ) we obtain the statements 1) and 2). Finally, (5.39) follows from the identity (2.45) for the triplet  $\Pi^n$  and Lemma 3.2, 2) applied to the solution  $v_\tau(\cdot, \lambda)$ .  $\square$

*Remark 5.10.* Let  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet for  $T_{max}$ , let  $\tau \in \tilde{R}^0(\mathcal{H}_b)$  be a boundary parameter given in the normalized form (2.19), (2.20) and let  $\bar{\Pi} = \{\mathcal{H}, \bar{\Gamma}_0, \bar{\Gamma}_1\}$  be a decomposing boundary triplet (5.40), (5.41) for  $T_{max}$ . It is easy to see that  $\Pi$  and  $\bar{\Pi}$  are connected by

$$\begin{pmatrix} \bar{\Gamma}_0 \\ \bar{\Gamma}_1 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix},$$

where  $X_j \in [H_0 \oplus \mathcal{H}_b]$  are defined as follows:

$$X_1 = \begin{pmatrix} I & 0 \\ 0 & C_0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 0 & -C_1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 0 & C_1 \end{pmatrix}, \quad X_4 = \begin{pmatrix} I & 0 \\ 0 & C_0 \end{pmatrix}.$$

Therefore according to [6] the Weyl functions  $M(\cdot)$  and  $\bar{M}(\cdot)$  of the triplets  $\Pi$  and  $\bar{\Pi}$  respectively are connected via

$$(5.43) \quad \bar{M}(\lambda) = (X_3 + X_4 M(\lambda))(X_1 + X_2 M(\lambda))^{-1}.$$

By using the block representation (5.9) of  $M(\lambda)$  one obtains  $(X_1 + X_2 M(\lambda))^{-1} =$

$$= \begin{pmatrix} I & 0 \\ -C_1 M_3 & C_0 - C_1 M_4 \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ (C_0 - C_1 M_4)^{-1} C_1 M_3 & (C_0 - C_1 M_4)^{-1} \end{pmatrix}$$

and (5.43), (5.42) imply that  $\bar{m}_0(\lambda)$  coincides with the right hand side of (5.10). This and the equality  $m_\tau(\lambda) = \bar{m}_0(\lambda)$  obtained in the proof of Proposition 5.9 yield (5.10). Thus, for canonical  $m$ -functions  $m_\tau(\cdot)$  formula (5.10) is a simple consequence of the relation (5.43) for Weyl functions.

Note that considerations in this remark are inspired by [6, Remark 86], where the Krein formula for resolvents was proved in a similar way.

## 6. PARTICULAR CASES

**6.1. Symmetric systems with minimal deficiency indices.** It follows from (3.13) that the minimally possible deficiency indices of  $T_{min}$  are

$$(6.1) \quad n_+ = \nu_+, \quad n_- = \nu_-$$

and (6.1) holds if and only if  $\nu_{b+} = \nu_{b-} = 0$  or, equivalently,  $[y, z]_b = 0$  for all  $y, z \in \text{dom } \mathcal{T}_{max}$ . This implies that the system (3.2) with minimal deficiency indices of  $T_{min}$  is in *Case 2* and by (3.17)  $\mathcal{H}_b = \widehat{\mathcal{H}}_b = \{0\}$ . Therefore in (3.35)

$$(6.2) \quad H'_0 = H, \quad \mathcal{H}'_2 = \widehat{H}$$

and the decomposing boundary triplet for  $T_{max}$  takes the form  $\Pi_- = \{H_0 \oplus H, \Gamma_0, \Gamma_1\}$ , where  $H_0$  is given by (3.11) and

$$(6.3) \quad \Gamma_0\{\widetilde{y}, \widetilde{f}\} = \{-\Gamma_{1a}y, i\widehat{\Gamma}_a y\} (\in H \oplus \widehat{H}), \quad \Gamma_1\{\widetilde{y}, \widetilde{f}\} = \Gamma_{0a}y (\in H), \quad \{\widetilde{y}, \widetilde{f}\} \in T_{max}.$$

In the case of minimal deficiency indices the symmetric extension  $T$  defined by (3.51) coincides with

$$(6.4) \quad A_0 (= \ker \Gamma_0) = \{\{\widetilde{y}, \widetilde{f}\} \in T_{max} : \Gamma_{1a}y = \widehat{\Gamma}_a y = 0\}$$

(c.f. (3.54)). The unique generalized resolvent  $R_0(\lambda)$  of  $A_0$  is of the form (4.49) and according to Theorem 4.6 it is given by the boundary value problem

$$(6.5) \quad Jy' - B(t)y = \lambda\Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},$$

$$(6.6) \quad \Gamma_{1a}y = 0, \quad \lambda \in \mathbb{C}_+; \quad \Gamma_{1a}y = 0, \quad \widehat{\Gamma}_a y = 0, \quad \lambda \in \mathbb{C}_-.$$

In view of Theorem 4.9 for each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists a unique operator solution  $v(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  of Eq. (3.3) such that

$$\Gamma_{1a}v(\lambda) = -P_H, \quad \lambda \in \mathbb{C}_+; \quad \Gamma_{1a}v(\lambda) = -P_H, \quad i\widehat{\Gamma}_a v(\lambda) = P_{\widehat{H}}, \quad \lambda \in \mathbb{C}_-,$$

and according to Definition 5.2 the  $m$ -function of the boundary value problem (6.5), (6.6) is given by

$$(6.7) \quad m(\lambda) = (\Gamma_{0a} + \widehat{\Gamma}_a)v(\lambda) + \frac{i}{2}P_{\widehat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In view of Proposition 5.3 the  $m$ -function  $m(\cdot)$  can be also defined by the relations

$$(6.8) \quad v(t, \lambda) := \varphi(t, \lambda)m(\lambda) + \psi(t, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R}; \quad i\widehat{\Gamma}_a v(\lambda) = P_{\widehat{H}}, \quad \lambda \in \mathbb{C}_-,$$

where  $\varphi(\cdot, \lambda)$  and  $\psi(\cdot, \lambda)$  are the solutions of Eq. (3.3) with the initial data (5.3).

Next, Proposition 4.8, 2) and Theorem 5.6 yield the following proposition.

**Proposition 6.1.** *Assume that  $T_{min}$  has minimal deficiency indices (6.1),  $\Pi_- = \{H_0 \oplus H, \Gamma_0, \Gamma_1\}$  is a decomposing boundary triplet (6.3) for  $T_{max}$ ,  $\gamma_-(\cdot)$  is the  $\gamma$ -field and*

$$(6.9) \quad M_+(\lambda) = (M(\lambda) : N_+(\lambda))^\top : H \rightarrow H \oplus \widehat{H}, \quad \lambda \in \mathbb{C}_+,$$

$$(6.10) \quad M_-(\lambda) = (M(\lambda) : N_-(\lambda)) : H \oplus \widehat{H} \rightarrow H, \quad \lambda \in \mathbb{C}_-,$$

are the corresponding Weyl functions. Then  $\gamma_-(\lambda) = \pi v(\lambda)$ ,  $\lambda \in \mathbb{C}_-$ , and the following equalities hold

$$(6.11) \quad m(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ N_+(\lambda) & \frac{i}{2}I_{\widehat{H}} \end{pmatrix} : \underbrace{H \oplus \widehat{H}}_{H_0} \rightarrow \underbrace{H \oplus \widehat{H}}_{H_0}, \quad \lambda \in \mathbb{C}_+$$

$$(6.12) \quad m(\lambda) = \begin{pmatrix} M(\lambda) & N_-(\lambda) \\ 0 & -\frac{i}{2}I_{\widehat{H}} \end{pmatrix} : \underbrace{H \oplus \widehat{H}}_{H_0} \rightarrow \underbrace{H \oplus \widehat{H}}_{H_0}, \quad \lambda \in \mathbb{C}_-.$$

Formulas (6.11) and (6.12) imply that the  $m$ -function  $m(\cdot)$  coincides with the function  $\mathcal{M}(\cdot)$  corresponding to the decomposing boundary triplet  $\Pi_-$  (see (2.38) and (2.39)).

Combining the latter statement of Proposition 6.1 with (2.40) and taking the equality  $\gamma_-(\lambda) = \pi v(\lambda)$  and Lemma 3.2, 2) into account we obtain the following corollary.

**Corollary 6.2.**  $m(\cdot)$  is a Nevanlinna operator function satisfying the identity

$$(6.13) \quad m(\mu) - m^*(\lambda) = (\mu - \overline{\lambda}) \int_{\mathcal{I}} v^*(t, \lambda) \Delta(t) v(t, \mu) dt, \quad \mu, \lambda \in \mathbb{C}_-.$$

*Remark 6.3.* It follows from (6.8) and (6.11) that the Weyl function (6.9) of the decomposing boundary triplet  $\Pi_-$  for  $T_{max}$  is defined by the relation

$$\varphi(t, \lambda) M_+(\lambda) + \chi(t, \lambda) \in \mathcal{L}_{\Delta}^2[H, \mathbb{H}], \quad \lambda \in \mathbb{C}_+,$$

where  $\chi(t, \lambda) (\in [H, \mathbb{H}])$  is a solution of Eq. (3.3) with the initial data (1.38). This and (1.37) imply that  $M_+(\lambda)$  coincides with the Titchmarsh - Weyl coefficient  $M_{TW}(\lambda)$  introduced in [20] for symmetric systems (3.2) with minimal deficiency indices  $n_{\pm}$  under the additional assumption that the operator  $X_a$  is of the special form (3.20). Observe also that the square matrix  $m(\lambda)$  defined by (6.11) appears in [20, p.34], where it forms the upper left block of the Nevanlinna matrix  $\widehat{M}(\lambda)$  (here  $\widehat{M}(\lambda)$  is the Titchmarsh - Weyl coefficient of the "doubled" system with equal deficiency indices).

**6.2. Symmetric systems with minimal equal deficiency indices.** It follows from (3.13) and (3.14) that the minimally possible equal deficiency indices of  $T_{min}$  are

$$(6.14) \quad n_+ = n_- = \nu_-$$

and the equalities (6.14) hold if and only if  $\nu_{b-} = 0$  and  $\nu_{b+} = \widehat{\nu}$ . Therefore by (3.17) and Proposition 3.4 the (ordinary) decomposing boundary triplet for  $T_{max}$  in the case (6.14) takes the form  $\Pi = \{H_0, \Gamma_0, \Gamma_1\}$  with

$$(6.15) \quad \Gamma_0\{\widetilde{y}, \widetilde{f}\} = (-\Gamma_{1a} + i(\widehat{\Gamma}_a - \widehat{\Gamma}_b))y, \quad \Gamma_1\{\widetilde{y}, \widetilde{f}\} = (\Gamma_{0a} + \frac{1}{2}(\widehat{\Gamma}_a + \widehat{\Gamma}_b))y, \quad \{\widetilde{y}, \widetilde{f}\} \in T_{max},$$

where  $\widehat{\Gamma}_b : \text{dom } \mathcal{T}_{max} \rightarrow \widehat{H}$  is a surjective linear mapping such that

$$[y, z]_b = i(\widehat{\Gamma}_b y, \widehat{\Gamma}_b z), \quad y, z \in \text{dom } \mathcal{T}_{max}.$$

Moreover, the extension (3.49) coincides with the self-adjoint extension

$$(6.16) \quad A_0 (= \ker \Gamma_0) = \{\{\widetilde{y}, \widetilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \widehat{\Gamma}_a y = \widehat{\Gamma}_b y\}$$

and the canonical resolvent  $R_0(\lambda) = (A_0 - \lambda)^{-1}$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , is defined by the boundary value problem (c.f. (4.5))

$$(6.17) \quad Jy' - B(t)y = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},$$

$$(6.18) \quad \Gamma_{1a}y = 0, \quad \widehat{\Gamma}_a y = \widehat{\Gamma}_b y, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

It follows from Theorem 4.5 that in the case (6.14) for each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists a unique operator solution  $v(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}]$  of Eq. (3.3) such that

$$\Gamma_1 v(\lambda) = -P_H, \quad i(\widehat{\Gamma}_a y - \widehat{\Gamma}_b)v(\lambda) = P_{\widehat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and according to Definition 5.2 and Proposition 5.3 the (canonical)  $m$ -function  $m(\cdot)$  of the boundary value problem (6.17), (6.18) is defined by (6.7) or, equivalently, by the relations

$$v(t, \lambda) := \varphi(t, \lambda)m(\lambda) + \psi(t, \lambda) \in \mathcal{L}_\Delta^2[H_0, \mathbb{H}], \quad i(\widehat{\Gamma}_a - \widehat{\Gamma}_b)v(\lambda) = P_{\widehat{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Finally, (4.19) and Corollary 5.5 yield the following proposition.

**Proposition 6.4.** *Let  $T_{min}$  has minimal equal deficiency indices (6.14), let  $\Pi = \{H_0, \Gamma_0, \Gamma_1\}$  be the decomposing boundary triplet (6.15) for  $T_{max}$ , let  $\gamma(\cdot)$  and  $M(\cdot)$  be the corresponding  $\gamma$ -field and Weyl function respectively and let  $m(\cdot)$  be the  $m$ -function of the boundary value problem (6.17), (6.18). Then*

$$\gamma(\lambda) = \pi v(\lambda), \quad M(\lambda) = m(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and the identity (5.39) holds with  $m_\tau(\lambda) = m(\lambda)$  and  $v_\tau(t, \lambda) = v(t, \lambda)$ .

**6.3. Hamiltonian systems.** Recall that the system (3.2) is called Hamiltonian if  $\nu_+ = \nu_- =: \nu$ , in which case the following assertions hold:

- 1)  $\widehat{H} = \{0\}$ , so that  $\mathbb{H} = H \oplus H$  (with  $\dim H = \nu$ ) and the signature operator (1.3) takes the form (1.5);
- 2) the deficiency indices of  $T_{min}$  are  $n_\pm = \nu + \nu_{b\pm}$  (c.f. (3.13));
- 3) the block representation (3.19) of the mapping  $\Gamma_a$  takes the form

$$\Gamma_a = \begin{pmatrix} \Gamma_{0a} \\ \Gamma_{1a} \end{pmatrix} : AC(\mathcal{I}; \mathbb{H}) \rightarrow H \oplus H.$$

In this subsection we let

$$\alpha = \text{sign}(\nu_{b+} - \nu_{b-}) = \text{sign}(n_+ - n_-).$$

Clearly, the Hamiltonian system (3.2) is in *Case 1* when  $\alpha \in \{0, 1\}$  and in *Case 3* when  $\alpha = -1$ . Moreover, in view of (3.14)  $n_+ = n_-$  if and only if

$$(6.19) \quad \nu_{b+} = \nu_{b-} =: \nu_b.$$

Assume that  $\mathcal{H}_b$  and  $\widehat{\mathcal{H}}_b$  are Hilbert spaces and  $\Gamma_b$  is a surjective linear map (3.15) such that (3.16) holds. Let  $\widetilde{\mathcal{H}}_b = \mathcal{H}_b \oplus \widehat{\mathcal{H}}_b$  and let  $\widetilde{\Gamma}_{0b} : \text{dom } T_{max} \rightarrow \widetilde{\mathcal{H}}_b$  be the mapping (3.40).

The following proposition is implied by Proposition 3.4.

**Proposition 6.5.** *Let in the case of the Hamiltonian system (3.2)  $\mathcal{H}_0 := H \oplus \widetilde{\mathcal{H}}_b$ ,  $\mathcal{H}_1 = H \oplus \mathcal{H}_b$  and let  $\Gamma_j : T_{max} \rightarrow \mathcal{H}_j$ ,  $j \in \{0, 1\}$ , be the linear maps given for  $\{\widetilde{y}, \widetilde{f}\} \in T_{max}$  by*

$$(6.20) \quad \Gamma_0\{\widetilde{y}, \widetilde{f}\} = \{-\Gamma_{1a}y, \widetilde{\Gamma}_{0b}y\} \in (H \oplus \widetilde{\mathcal{H}}_b), \quad \Gamma_1\{\widetilde{y}, \widetilde{f}\} = \{\Gamma_{0a}y, -\Gamma_{1b}y\} \in (H \oplus \mathcal{H}_b).$$

Then the collection  $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  is a decomposing boundary triplet for  $T_{max}$ .

If in addition  $n_+ = n_-$ , then  $\Pi_\alpha$  turns into the ordinary boundary triplet  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ , where  $\mathcal{H} = H \oplus \mathcal{H}_b$  and the operators  $\Gamma_0$  and  $\Gamma_1$  are given by (6.20) with  $\widetilde{\mathcal{H}}_b = \mathcal{H}_b$  and  $\widetilde{\Gamma}_{0b} = \Gamma_{0b}$ .

It follows from Propositions 3.7 and 3.9 that in the case of the Hamiltonian system the equality

$$(6.21) \quad T = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0, \tilde{\Gamma}_{0b}y = \Gamma_{1b}y = 0\}$$

defines a symmetric extension  $T$  of  $T_{min}$  with the deficiency indices  $n_{\pm}(T) = \nu_{b\pm}$ . If in addition  $n_+ = n_-$ , then in (6.21)  $\tilde{\Gamma}_{0b} = \Gamma_{0b}$  and  $T$  has equal deficiency indices  $n_+(T) = n_-(T) = \nu_b$ .

Theorems 4.1 and 4.10 yield the following theorem.

**Theorem 6.6.** *Let in the case of the Hamiltonian system (3.2)  $\Pi_{\alpha} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$  be a decomposing boundary triplet (6.20) for  $T_{max}$ . If  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_{\alpha}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  is a collection (2.3), then for every  $f \in \mathcal{L}_{\Delta}^2(\mathcal{I})$  the boundary value problem*

$$(6.22) \quad Jy' - B(t)y = \lambda\Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},$$

$$(6.23) \quad \Gamma_{1a}y = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

$$(6.24) \quad C_0(\lambda)\tilde{\Gamma}_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_+; \quad D_0(\lambda)\tilde{\Gamma}_{0b}y + D_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_-,$$

has a unique solution  $y(t, \lambda) = y_f(t, \lambda)$  and the equality (4.4) defines a generalized resolvent  $R(\lambda) =: R_{\tau}(\lambda)$  of  $T$  (see (6.21)). Conversely, for each generalized resolvent  $R(\lambda)$  of  $T$  there exists a unique  $\tau \in \tilde{R}_{\alpha}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  such that  $R(\lambda) = R_{\tau}(\lambda)$ .

If in addition  $n_+ = n_-$  and  $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$  is an ordinary decomposing boundary triplet for  $T_{max}$ , then the statements of the theorem hold with the Nevanlinna operator pairs  $\tau \in \tilde{R}(\mathcal{H}_b)$  in the form (2.16) and the boundary conditions

$$(6.25) \quad \Gamma_{1a}y = 0, \quad C_0(\lambda)\Gamma_{0b}y + C_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

instead of (6.23) and (6.24). In this case  $R_{\tau}(\lambda)$  is a canonical resolvent of  $T$  if and only if  $\tau \in \tilde{R}^0(\mathcal{H}_b)$ .

For Hamiltonian systems the operator solution  $v_{\tau}(\cdot, \lambda)$  takes on values in  $[H, \mathbb{H}]$  in place of  $[H_0, \mathbb{H}]$  for general systems. More precisely, the following theorem is implied by Theorems 4.5 and 4.13.

**Theorem 6.7.** *Let the assumptions of Theorem 6.6 be satisfied. Then for each collection  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_{\alpha}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  defined by (2.3) there exists a unique solution  $v_{\tau}(\cdot, \lambda) \in \mathcal{L}_{\Delta}^2[H, \mathbb{H}]$  of Eq. (3.3) such that*

$$(6.26) \quad \Gamma_{1a}v_{\tau}(\lambda) = -I_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(6.27) \quad C_0(\lambda)\tilde{\Gamma}_{0b}v_{\tau}(\lambda) + C_1(\lambda)\Gamma_{1b}v_{\tau}(\lambda) = 0, \quad \lambda \in \mathbb{C}_+,$$

$$(6.28) \quad D_0(\lambda)\tilde{\Gamma}_{0b}v_{\tau}(\lambda) + D_1(\lambda)\Gamma_{1b}v_{\tau}(\lambda) = 0, \quad \lambda \in \mathbb{C}_-.$$

If  $n_+ = n_-$ , then  $\tau \in \tilde{R}(\mathcal{H}_b)$  is given by (2.16) and the conditions (6.26)–(6.28) take the form

$$(6.29) \quad \Gamma_{1a}v_{\tau}(\lambda) = -I_H, \quad C_0(\lambda)\Gamma_{0b}v_{\tau}(\lambda) + C_1(\lambda)\Gamma_{1b}v_{\tau}(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Next, a boundary parameter in the sense of Definition 5.1 is a collection  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_{\alpha}(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  of operator pairs (2.3). Moreover, in the case  $n_+ = n_-$  the boundary parameter is an operator pair  $\tau \in \tilde{R}(\mathcal{H}_b)$  given by (2.16).

For Hamiltonian systems Definition 5.2 and Proposition 5.3 take the following form.



**Definition 6.8.** The operator function  $m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [H]$  defined by

$$m_\tau(\lambda) = \Gamma_{0a} v_\tau(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

is called the  $m$ -function corresponding to the boundary parameter  $\tau$  or, equivalently, to the boundary value problem (6.22)–(6.24).

If  $n_+ = n_-$ , then  $m_\tau(\cdot)$  corresponds to the problem (6.22), (6.25). In this case the  $m$ -function  $m_\tau(\cdot)$  is called canonical if  $\tau \in \tilde{R}^0(\mathcal{H}_b)$ .

**Proposition 6.9.** The  $m$ -function  $m_\tau(\cdot)$  is a unique  $[H]$ -valued function such that

$$(6.30) \quad v_\tau(t, \lambda) := \varphi(t, \lambda) m_\tau(\lambda) + \psi(t, \lambda) \in \mathcal{L}_\Delta^2[H, \mathbb{H}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and the boundary conditions (6.27) and (6.28) are satisfied. If, in addition,  $n_+ = n_-$  and  $\tau \in \tilde{R}^0(\mathcal{H}_b)$  is given by (2.19), then the conditions (6.27) and (6.28) take the form

$$(6.31) \quad C_0 \Gamma_{0b} v_\tau(\lambda) + C_1 \Gamma_{1b} v_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In formula (6.30)  $\varphi(\cdot, \lambda)$  and  $\psi(\cdot, \lambda)$  are the operator solutions of Eq. (3.3) with values in  $[H, H \oplus H]$  and such that

$$\varphi_a(\lambda) = \begin{pmatrix} I_H \\ 0 \end{pmatrix} : H \rightarrow H \oplus H, \quad \psi_a(\lambda) = \begin{pmatrix} 0 \\ -I_H \end{pmatrix} : H \rightarrow H \oplus H, \quad \lambda \in \mathbb{C}.$$

The following theorem is implied by Theorems 5.4, 5.7 and Corollary 5.5.

**Theorem 6.10.** Let the conditions of Theorem 6.6 be satisfied and let  $\tau_0 = \{\tau_+, \tau_-\}$  be a boundary parameter (2.3) defined by (4.6) in the case  $n_+ > n_-$  and by (4.91) in the case  $n_+ < n_-$ . Assume also that the Weyl functions  $M_\pm(\cdot)$  have the block representations:

— in the case  $n_+ > n_-$

$$M_+(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix} : H \oplus \tilde{\mathcal{H}}_b \rightarrow H \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_+;$$

— in the case  $n_+ < n_-$

$$M_-(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2-}(\lambda) \\ M_{3-}(\lambda) & M_{4-}(\lambda) \end{pmatrix} : H \oplus \tilde{\mathcal{H}}_b \rightarrow H \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_-.$$

Moreover, let  $\tau = \{\tau_+, \tau_-\}$  be a boundary parameter (2.3). Then: 1)  $m_0(\lambda) = m_{\tau_0}(\lambda)$ ; 2) in the case  $n_+ > n_-$  the equality (5.5) holds; 3) in the case  $n_+ < n_-$  the equality

$$m_\tau(\lambda) = m_0(\lambda) + M_{2-}(\lambda)(D_0(\lambda) - D_1(\lambda)M_{4-}(\lambda))^{-1}D_1(\lambda)M_{3-}(\lambda), \quad \lambda \in \mathbb{C}_-,$$

is valid. Moreover, if  $n_+ = n_-$  and the Weyl function  $M(\lambda)$  has the block representation (5.9) with  $H$  instead of  $H_0$ , then for every boundary parameter  $\tau$  in the form (2.16) the equality (5.10) is valid.

For the Hamiltonian system the minimally possible deficiency indices of  $T_{min}$  are

$$(6.32) \quad n_+ = n_- = \nu(= \dim H),$$

in which case the boundary triplet (6.20) for  $T_{max}$  takes the form  $\Pi = \{H, \Gamma_0, \Gamma_1\}$  with

$$(6.33) \quad \Gamma_0\{\tilde{y}, \tilde{f}\} = -\Gamma_{1a}y, \quad \Gamma_1\{\tilde{y}, \tilde{f}\} = \Gamma_{0a}y, \quad \{\tilde{y}, \tilde{f}\} \in T_{max},$$

and the extension (6.21) turns into the self-adjoint extension

$$A_0(= \ker \Gamma_0) = \{\{\tilde{y}, \tilde{f}\} \in T_{max} : \Gamma_{1a}y = 0\}.$$

Moreover, the resolvent  $R(\lambda) = (A_0 - \lambda)^{-1}$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , of this extension is defined by the boundary value problem (6.22), (6.23).

It follows from Theorem 6.7 that in the case (6.32) for each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists a unique operator solution  $v(\cdot, \lambda) \in \mathcal{L}_\Delta^2[H, \mathbb{H}]$  of Eq. (3.3) such that  $\Gamma_{1a}v(\lambda) = -I_H$ . The  $m$ -function of the problem (6.22), (6.23) is defined by  $m(\lambda) = \Gamma_{0a}v(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , or, equivalently, by the relation

$$v(t, \lambda) := \varphi(t, \lambda)m(\lambda) + \psi(t, \lambda) \in \mathcal{L}_\Delta^2[H, \mathbb{H}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Note in conclusion, that according to Proposition 6.4  $m(\cdot)$  is the Weyl function of the boundary triplet (6.33).

*Remark 6.11.* Assume that  $n_+ = n_-$  and the operators  $\Gamma_{0b}$  and  $\Gamma_{1b}$  in (6.20) are defined by (3.26). Moreover, let  $\tau \in \widetilde{R}^0(\mathbb{C}^{\nu_b})$  be a self-adjoint boundary parameter (2.19). Then by using the matrix representation of  $C_0$  and  $C_1$  one can express the boundary condition (6.31) as

$$[v_\tau(\cdot, \lambda)h, \chi_j]_b = 0, \quad h \in H, \quad j = 1 \div \nu_b,$$

where  $\chi_j \in \text{dom } \mathcal{T}_{max}$  are linear combinations of  $\psi_k$  and  $\theta_k$ . This and Proposition 6.9 imply that the canonical  $m$ -function  $m_\tau(\cdot)$  is the Titchmarsh - Weyl coefficient of the Hamiltonian system in the sense of [19].

## 7. APPLICATIONS TO DIFFERENTIAL OPERATORS

In this section we apply the above results to operators generated by a differential expression  $l[y]$  of an odd order  $r = 2n + 1$  defined on an interval  $\mathcal{I} = [a, b)$  ( $-\infty < a < b \leq \infty$ ) with the regular endpoint  $a$ . Such an expression is of the form [41]

$$(7.1) \quad l[y] = \frac{1}{w} \left\{ \sum_{k=0}^n (-1)^k \left( \frac{i}{2} [(q_{n-k}y^{(k)})^{(k+1)} + (q_{n-k}y^{(k+1)})^{(k)}] + (p_{n-k}y^{(k)})^{(k)} \right) \right\},$$

where  $p_j(\cdot)$ ,  $q_j(\cdot)$  and  $w(\cdot)$  are real valued functions on  $\mathcal{I}$  such that : 1)  $p_j(\cdot)$  and  $q_j(\cdot)$  are smooth enough and  $q_0(t) > 0$ ,  $t \in \mathcal{I}$ ; 2)  $w(\cdot) \in L_1([a, \beta])$  for each  $\beta \in \mathcal{I}$  and  $w(t) > 0$  a.e. on  $\mathcal{I}$ . Denote by  $y^{[k]}(\cdot)$ ,  $k = 0 \div 2n + 1$  the quasi-derivatives of a complex-valued function  $y(\cdot) \in Ac(\mathcal{I})$  and let  $\text{dom } l$  be the set of functions  $y(\cdot)$  for which  $l[y] := y^{[2n+1]}$  makes sense [41, 24, 40]. For each  $y \in \text{dom } l$  we let

$$y^{(1)}(t) := \{y^{[k-1]}(t)\}_{k=1}^n (\in \mathbb{C}^n), \quad y^{(2)}(t) := \{y^{[2n-k+1]}(t)\}_{k=1}^n (\in \mathbb{C}^n),$$

$$\mathbf{y}(t) = \{y^{(1)}(t), \widehat{y}(t), y^{(2)}(t)\} (\in \mathbb{C}^{2n+1}),$$

where  $\widehat{y}(t) = q_0^{\frac{1}{2}}(t)y^{(n)}(t)$ . Moreover, for each  $m$ -component operator solution

$$(7.2) \quad Y(t, \lambda) = (y_1(t, \lambda) : y_2(t, \lambda) : \dots : y_m(t, \lambda)) : \mathbb{C}^m \rightarrow \mathbb{C}$$

of the differential equation

$$(7.3) \quad l[y] = \lambda y \quad (\lambda \in \mathbb{C})$$

we put

$$Y^{(j)}(t, \lambda) = (y_1^{(j)}(t, \lambda) : y_2^{(j)}(t, \lambda) : \dots : y_m^{(j)}(t, \lambda)) : \mathbb{C}^m \rightarrow \mathbb{C}^n, \quad j \in \{1, 2\},$$

$$\widehat{Y}(t, \lambda) = (\widehat{y}_1(t, \lambda) : \widehat{y}_2(t, \lambda) : \dots : \widehat{y}_m(t, \lambda)) : \mathbb{C}^m \rightarrow \mathbb{C},$$

$$\mathbf{Y}(t, \lambda) = (Y^{(1)}(t, \lambda) : \widehat{Y}(t, \lambda) : Y^{(2)}(t, \lambda))^\top : \mathbb{C}^m \rightarrow \mathbb{C}^{2n+1}, \quad t \in \mathcal{I}.$$

Next assume that  $L_r^2(\mathcal{I})$  is the Hilbert space of all complex-valued Borel functions on  $\mathcal{I}$  such that  $\int_{\mathcal{I}} r(t)|f(t)|^2 dt < \infty$ . It is known [41] that the expression (7.1) generates in  $L_r^2(\mathcal{I})$  the maximal operator  $L$  and the minimal operator  $L_0$ . Moreover,  $L_0$  is a closed densely defined symmetric operator and  $L_0^* = L$ .

By using the results of [24] one can easily prove the following assertion.

**Assertion 7.1.** *Let  $l[y]$  be the expression (7.1) and let*

$$J_0 = \begin{pmatrix} 0 & 0 & -I_n \\ 0 & iI_1 & 0 \\ I_n & 0 & 0 \end{pmatrix} \in [\mathbb{C}^n \oplus \mathbb{C} \oplus \mathbb{C}^n], \quad \Delta_0(t) = \begin{pmatrix} w(t) & 0 \\ 0 & 0 \end{pmatrix} \in [\mathbb{C} \oplus \mathbb{C}^{2n}].$$

*Then there exists a continuous operator function  $B_0(t) = B_0^*(t) \in [\mathbb{C}^{2n+1}]$  (defined in terms of  $p_j$  and  $q_j$ ) such that:*

*1) a complex-valued function  $y(\cdot, \lambda)$  (operator function  $Y(\cdot, \lambda)$  of the form (7.2)) is a solution of Eq. (7.3) if and only if  $\mathbf{y}(\cdot, \lambda)$  (resp.  $\mathbf{Y}(\cdot, \lambda)$ ) is a solution of the symmetric system*

$$(7.4) \quad J_0 y' - B_0(t)y = \lambda \Delta_0(t)y, \quad t \in \mathcal{I};$$

*2) the equality*

$$(7.5) \quad V\{y, f\} = \{\mathbf{y}, \dot{f}\}, \quad \{y, f\} \in \text{gr } L,$$

*with  $\dot{f}(t) = \{f(t), 0, \dots, 0\} \in \mathbb{C}^{2n+1}$  defines a unitary operator  $V : \text{gr } L \rightarrow \mathcal{T}_{\max}$ , where  $\mathcal{T}_{\max}$  is the maximal relation in  $\mathcal{L}_{\Delta_0}^2(\mathcal{I})$  for the system*

$$(7.6) \quad J_0 y' - B_0(t)y = \Delta_0(t)f(t), \quad t \in \mathcal{I}.$$

*Moreover,  $V \text{ gr } L_0 = \mathcal{T}_{\min}$ , where  $\mathcal{T}_{\min}$  is the minimal relation for the system (7.6).*

Assertion 7.1 enables us to identify all the objects related to the expression (7.1) with similar objects for the system (7.6). In particular, we assume that: 1)  $\nu_{b+}$  and  $\nu_{b-}$  are indices of inertia of the bilinear form (3.6) for the system (7.6); 2) the linear map  $\Gamma_b$  in (3.15) is defined on  $\text{dom } L$ , so that  $\Gamma_{0b}y$ ,  $\Gamma_{1b}y$  and  $\hat{\Gamma}_b y$  are the singular boundary values of a function  $y \in \text{dom } L$  and its quasi-derivatives (c.f. Remark 3.3). Moreover, let  $X_a \in [\mathbb{C}^{2n+1}]$  be the operator such that  $X_a^* J_0 X_a = J_0$  and let

$$\Gamma_a = (\Gamma_{0a} : \hat{\Gamma}_a : \Gamma_{1a})^\top : \text{dom } l \rightarrow \mathbb{C}^n \oplus \mathbb{C} \oplus \mathbb{C}^n$$

be the linear map given by  $\Gamma_a y = X_a \mathbf{y}(a)$ ,  $y \in \text{dom } l$ .

Clearly, for the system (7.6) one has  $\nu_- - \nu_+ = 1$ . Hence this system is either in *Case 1* or in *Case 3* and the reasonings in Subsection 3.3 take the following form:

1) *Case 1:*  $\nu_{b+} - \nu_{b-} \geq 1$  or, equivalently,  $n_-(L_0) \leq n_+(L_0)$ . In this case we put  $d = \nu_{b+} - \nu_{b-} - 1$ ,  $\mathcal{H}_b = \mathbb{C}^{\nu_{b-}}$  and  $\hat{\mathcal{H}}_b = \mathbb{C}^d \oplus \mathbb{C}$  (c.f. (3.28)), so that the operator  $\hat{\Gamma}_b$  can be represented as

$$\hat{\Gamma}_b = (\hat{\Gamma}'_b : \hat{\Gamma}''_b)^\top : \text{dom } L \rightarrow \mathbb{C}^d \oplus \mathbb{C}.$$

This implies that  $\tilde{\mathcal{H}}_b = \mathbb{C}^{\nu_{b-}} \oplus \mathbb{C}^d$  and by (3.29)  $\tilde{\Gamma}_{0b} = (\Gamma_{0b} : \hat{\Gamma}'_b)^\top$ .

2) *Case 3:*  $\nu_{b+} - \nu_{b-} \leq 0$  or, equivalently,  $n_+(L_0) < n_-(L_0)$ . We put  $d' = \nu_{b-} - \nu_{b+}$ ,  $\mathcal{H}_b = \mathbb{C}^{\nu_{b+}}$  and  $\hat{\mathcal{H}}_b = \mathbb{C}^{d'}$  (c.f. (3.39)). Then  $\tilde{\mathcal{H}}_b = \mathbb{C}^{\nu_{b+}} \oplus \mathbb{C}^{d'} (= \mathbb{C}^{\nu_{b-}})$  and in view of (3.40) one has

$$\tilde{\Gamma}_{0b} = (\Gamma_{0b} : \hat{\Gamma}_b)^\top : \text{dom } L \rightarrow \mathbb{C}^{\nu_{b+}} \oplus \mathbb{C}^{d'}.$$

Now by using Assertion 7.1 one can easily reformulate all the previous results for symmetric systems (3.2) in terms of the expression (7.1). For example Theorems 4.5, 4.9 and 4.13 take the following form.

**Theorem 7.2.** *Let  $\tau = \{\tau_+, \tau_-\} \in \tilde{R}_\alpha(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  be a collection of operator pairs (2.3) with  $\alpha = +1$  in the case  $n_-(L_0) \leq n_+(L_0)$  and  $\alpha = -1$  in the case  $n_+(L_0) < n_-(L_0)$ . Then for each  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  there exists a unique  $(n+1)$ -component operator solution*

$$(7.7) \quad v_\tau(t, \lambda) = (v_1(t, \lambda) : v_2(t, \lambda) : \dots : v_n(t, \lambda) : v_{n+1}(t, \lambda)) : \mathbb{C}^n \oplus \mathbb{C} \rightarrow \mathbb{C}$$

of Eq. (7.3) such that  $v_j(\cdot, \lambda) \in L_r^2(\mathcal{I})$ ,  $j = 1 \div (n+1)$ , and the following boundary conditions are satisfied:

1) in the case  $n_-(L_0) \leq n_+(L_0)$

$$(7.8) \quad \Gamma_{1a} v_\tau(\lambda) = (-I_{\mathbb{C}^n} : 0) : \mathbb{C}^n \oplus \mathbb{C} \rightarrow \mathbb{C}^n, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(7.9) \quad i(\hat{\Gamma}_a - \hat{\Gamma}_b'') v_\tau(\lambda) = (0_{\mathbb{C}^n} : I_{\mathbb{C}}) : \mathbb{C}^n \oplus \mathbb{C} \rightarrow \mathbb{C}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$(7.10) \quad C_0(\lambda) \tilde{\Gamma}_{0b} v_\tau(\lambda) + C_1(\lambda) \Gamma_{1b} v_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C}_+,$$

$$(7.11) \quad D_0(\lambda) \tilde{\Gamma}_{0b} v_\tau(\lambda) + D_1(\lambda) \Gamma_{1b} v_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C}_-.$$

2) in the case  $n_+(L_0) < n_-(L_0)$  – the conditions (7.8), (7.10), (7.11) and

$$(7.12) \quad i\hat{\Gamma}_a v_\tau(\lambda) = (0_{\mathbb{C}^n} : I_{\mathbb{C}}) : \mathbb{C}^n \oplus \mathbb{C} \rightarrow \mathbb{C}, \quad \lambda \in \mathbb{C}_-.$$

Here the linear map  $v_\tau(\lambda) : \mathbb{C}^n \oplus \mathbb{C} \rightarrow \mathfrak{N}_\lambda(L_0)$  is given by

$$(v_\tau(\lambda)h)(t) = v_\tau(t, \lambda)h = \sum_{j=1}^{n+1} v_j(t, \lambda)h_j, \quad h = \{h_1, h_2, \dots, h_n, h_{n+1}\} \in \mathbb{C}^n \oplus \mathbb{C},$$

so that  $\Gamma_a v_\tau(\lambda) = X_a \mathbf{v}_\tau(a, \lambda)$ .

Next, the  $m$ -function  $m_\tau(\cdot)$  of the expression  $l[y]$  corresponding to the boundary parameter  $\tau \in \tilde{R}_\alpha(\tilde{\mathcal{H}}_b, \mathcal{H}_b)$  (with the same  $\alpha$  as in Theorem 7.2) is defined as the  $m$ -function of the system (7.6). In view of Proposition 5.3 this means that  $m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\mathbb{C}^{n+1}]$  is a unique operator function such that for every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  the  $(n+1)$ -component operator solution (7.7) of Eq. (7.3) given by

$$v_\tau(t, \lambda) := Y_1(t, \lambda)m_\tau(\lambda) + Y_2(t, \lambda)$$

possesses the following properties: 1)  $v_j(\cdot, \lambda) \in L_r^2(\mathcal{I})$  for all  $j = 1 \div (n+1)$ , 2)  $v_\tau(\cdot, \lambda)$  satisfies the boundary conditions (7.9)-(7.11) in the case  $n_-(L_0) \leq n_+(L_0)$  and (7.10)-(7.12) in the case  $n_+(L_0) < n_-(L_0)$ . Here  $Y_1(t, \lambda)$  and  $Y_2(t, \lambda)$  are the  $(n+1)$ -component operator

solutions of Eq. (7.3) with the initial data  $X_a \mathbf{Y}_1(a, \lambda) = \begin{pmatrix} I_n & 0 \\ 0 & I_1 \\ 0 & 0 \end{pmatrix} \in [\mathbb{C}_n \oplus \mathbb{C}, \mathbb{C}_n \oplus \mathbb{C} \oplus \mathbb{C}^n]$

and  $X_a \mathbf{Y}_2(a, \lambda) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{i}{2}I_1 \\ -I_n & 0 \end{pmatrix} \in [\mathbb{C}_n \oplus \mathbb{C}, \mathbb{C}_n \oplus \mathbb{C} \oplus \mathbb{C}^n]$ .

According to results in Section 5  $m_\tau(\cdot)$  is a Nevanlinna operator function, which in the case  $n_+(L_0) < n_-(L_0)$  has the triangular form (5.20) and (5.21) with  $H = \mathbb{C}^n$  and  $\hat{H} = \mathbb{C}$ .

The reformulations of other results in Sections 5 and 6 to the case of the  $m$ -functions of the differential expression (7.1) are left to the reader.

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